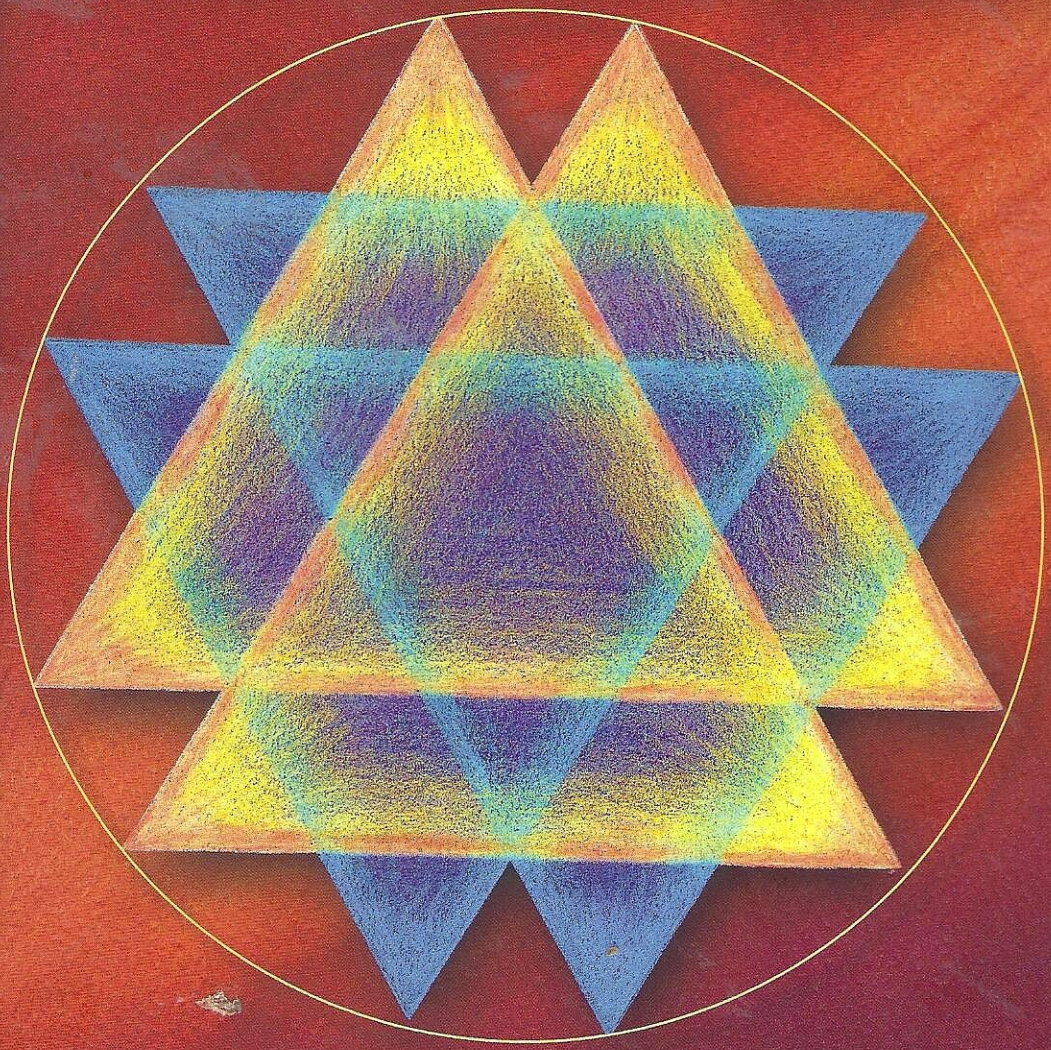
THE ZEN OF MAGIC SQUARES, CIRCLES, AND STARS



CLIFFORD A. PICKOVER

THE ZEN OF MAGIC SQUARES

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AN EXHIBITION OF SURPRISING STRUCTURES ACROSS DIMENSIONS

CLIFFORD A. PICKOVER

This is the most comprehensive and exciting book in many decades on magic squares—arrays filled with numbers or letters in particular arrangements. Clifford Pickover offers a colorful history of these and similar structures, their construction, and classification along with a remarkable variety of newly discovered objects ranging from ornate inlaid magic cubes to hypercubes. He explains why Chinese emperors, Babylonian astrologer-priests, prehistoric cave people in France, and ancient Mayans of the Yucatan were convinced that magic squares held the secret of the universe.

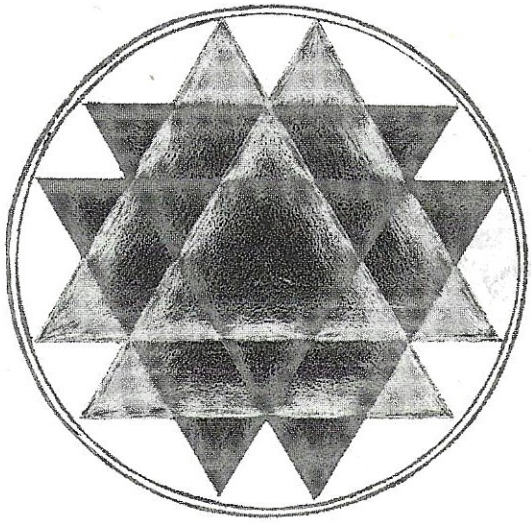
Since the dawn of civilization, writes Pickover, humans have invoked such patterns to ward off evil and bring good fortune. Yet who would have guessed that in the twenty-first century mathematicians would be studying magic squares so immense and in so many dimensions that the objects defy ordinary human contemplation and visualization?

Illustrated examples occur throughout, with some patterns from the author's own experiments. The tesseracts, circles, spheres, and stars that he presents perfectly convey the age-old devotion of the math-minded to this Zenlike quest. Number lovers, puzzle aficionados, and math enthusiasts will treasure this rich and lively encyclopedia of one of the few areas of mathematics where the contributions of even nonspecialists count.

THE ZEN OF MAGIC SQUARES, CIRCLES, AND STARS

An Exhibition of Surprising Structures across Dimensions

Clifford A. Pickover



Princeton University Press
Princeton and Oxford

Copyright © 2002 by Clifford A. Pickover

Published by Princeton University Press, 41 William Street,
Princeton, New Jersey 08540

In the United Kingdom: Princeton University Press, 3 Market Place, Woodstock,
Oxfordshire OX20 1SY

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Library of Congress Cataloging-in-Publication Data

Pickover, Clifford A.
The zen of magic squares, circles, and stars : an exhibition of surprising structures
across dimensions / Clifford A. Pickover.
p. cm
Includes bibliographical references and index.
ISBN 0-691-07041-5 (acid-free paper)
1. Magic squares. 2. Mathematical recreations. I. Title.
QA165.P53 2002
511'.64–dc21 2001027848

British Library Cataloging-in-Publication Data is available

This book has been composed in Baskerville BE and Gill Sans.

Printed on acid-free paper ∞

www.pup.princeton.edu

Printed in the United States of America
10 9 8 7 6 5 4 3 2 1

The peculiar interest of magic squares lies in the fact
that they possess the charm of mystery.
They appear to betray some hidden intelligence
which by a preconceived plan produces
the impression of intentional design,
a phenomenon which finds its
close analogue in nature.

—Paul Carus, in W. S. Andrews's *Magic Squares and Cubes*

The mathematical phenomenon always develops
out of simple arithmetic, so useful in everyday life,
out of numbers, those weapons of the gods:
the gods are there, behind the wall,
at play with numbers.

—Le Corbusier, *The Modulor*

To study magic squares is to study the self.
To study the self is to forget the self.
To forget the self is to be enlightened.

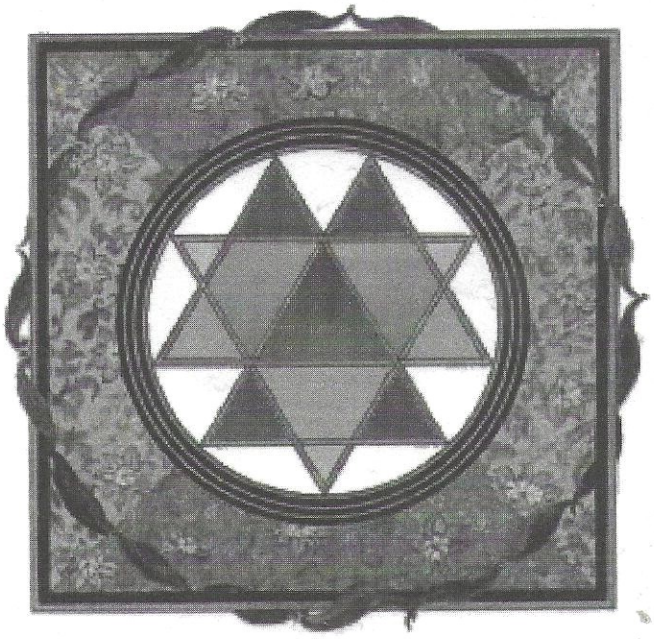
—Abhinavagupta Isvarapratyabhijna[1]

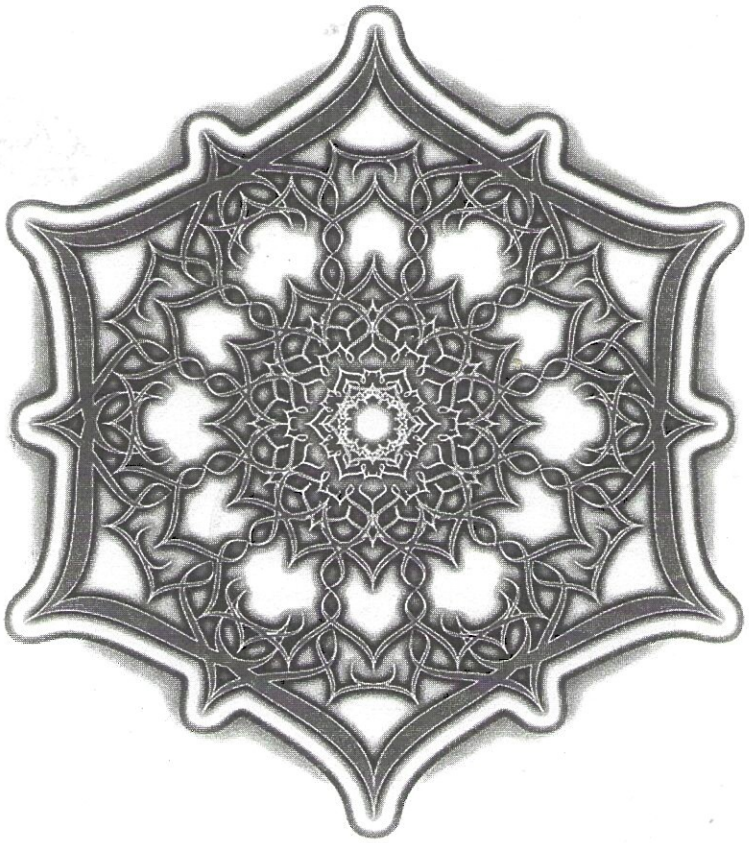
The magic square is the hammer that
shatters the ice of our unconscious.

—Qingfu Chuzhen[2]

Translation: the Zen of magic squares. *The phrase "magic squares" is written literally as "square puzzles." On its own, this Chinese word for puzzle refers to the location of soldiers and weapons on a battlefield as described in Sun Zi's* Art of War. *However, together with the Chinese word for "square," the phrase denotes "magic squares." The small square at the bottom encloses the name of the calligrapher, Siu-Leung Lee.*

This book is dedicated not to a person
but to a meditative aid,
the Durga Yantra,
to which numbers
can be applied
in magic
ways.





Introduction

Magic squares contain a lesson of great value in being a palpable instance of the symmetry of mathematics, throwing thereby a clear light upon the order that pervades the universe wherever we turn, in the infinitesimally small interrelations of atoms as well as in the immeasurable domain of the starry heavens.

–Paul Carus, in W. S. Andrews's *Magic Squares and Cubes*

What Is a Magic Square?

A *magic square* is a square matrix drawn as a checkerboard filled with numbers or letters in particular arrangements. Mathematicians are most interested in *arithmetic* squares consisting of N^2 boxes, called *cells*, filled with integers that are all different. Such an array of numbers is called a magic square if the sums of the numbers in the horizontal rows, vertical columns, and main diagonals are all equal. If the integers in a magic square are the consecutive numbers from 1 to N^2, the square is said to be of the Nth order, and the *magic number*, or sum of each row, is a constant symbolized as $\mathcal{S}$:

$$\mathcal{S} = \frac{N(N^2+1)}{(2)}$$

(The magic number is sometimes referred to as the *magic sum* or *magic constant*.) To derive this expression for $\mathcal{S}$, recall that the sum of the first m numbers in the arithmetic series $1 + 2 + 3 + \cdots + m$ is equal to $m(m+1)/2$. In our case $m = N^2$ because we are interested in the sum of all numbers in the magic square. This means that the sum of the numbers in a magic square is $N^2(N^2+1)/2 = (N^4 + N^2)/2$. To get $\mathcal{S}$, we divide this result by N, which gives the sum for each of the N rows and N columns.

A few examples will help demystify these mathematical definitions. The simplest magic square possible is one of the third order, with 3×3 cells containing the integers 1 through 9, and with the magic sum 15 along the three rows, three columns, and two diagonals. In some sense, only one unique arrangement of digits, and its mirror image, is possible for a third-order square:

4	9	2
3	5	7
8	1	6

Third-order magic square

2	9	4
7	5	3
6	1	8

Mirror image

Here, $N = 3$ because there are three rows and three columns, and the magic sum $\mathcal{S}$ is 15 because the numbers in the rows, columns, and two diagonals sum to 15. For example, if you look at the square on the left, you will see that the sum of the numbers in the first row is $4 + 9 + 2 = 15$, the sum of the numbers in the first column is $4 + 3 + 8 = 15$, one of the diagonal sums is $4 + 5 + 6 = 15$, and so forth. We can also use the magic sum formula to compute the magic sum: $3(3^2 + 1)/2 = 15$. Notice that the mirror image is also a magic square. By rotating the square four times by 90 degrees, you can produce eight third-order magic squares.

To quickly understand why the number 5 must be at the center of the third-order square, consider all the ways in which the magic constant, 15, can be partitioned into a triplet of distinct positive integers:

$$9 + \mathbf{5} + 1, \quad 9 + 4 + 2, \quad 8 + 6 + 1, \quad 8 + \mathbf{5} + 2$$

$$8 + 4 + 3, \quad 7 + 6 + 2, \quad 7 + \mathbf{5} + 3, \quad 6 + \mathbf{5} + 4$$

In a third-order magic square, each of the three rows, three columns, and two diagonals must sum to 15. In other words, there must be eight sets of numbers that sum to 15, and these are repre-

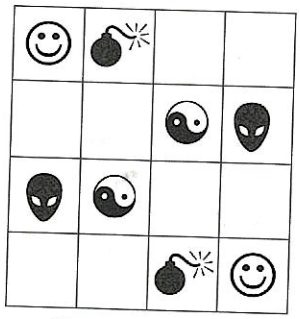
sented by the eight previous sums. Since the center number belongs to one row, one column, and two diagonals, it must appear in four of the eight equations. The only such digit is 5. This means that 5 must be in the center of the 3×3 square.

Here is an example of a fourth-order magic square:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Fourth-order magic square

Cells placed at equal distances from the center cell and on opposite ends of an imaginary line through the center are called *skew-related cells.* A few examples of skew-related cells are shown in the following diagram. For example, the cells with the two smiley faces are skew related, the two alien faces are skew related, and so forth.



Skew-related cells

If we do not consider reflections and rotations, there is only one third-order magic square. As the order N increases,[1] the number of magic squares increases rapidly:

N	*Number of Different Magic Squares*
1	1
2	0
3	1
4	880
5	275,305,224

As far back as 1693, the 880 different fourth-order magic squares were published posthumously by French mathematician Bernard Frénicle de Bessy (1602–1675). Frénicle, one of the great magic square researchers of all time, was an eminent amateur mathematician working in Paris during the great period of French mathematics in the seventeenth century.[2] His magic square research was often carried out by a "method of exhaustion" rather than by deep mathematical theory, and his lists of solutions for fourth-order squares was published as an important part of a substantial treatise entitled *Des Quarrez ou Tables Magiques*, one of four treatises collected by Phillipe de la Hire (1640–1718), a French geometer who was also interested in magic square construction. In 1731, the treatise was republished as *Divers Ouvrages de Mathématique par Messieurs de l'Académie des Sciences* in the Hague, and it is this publication that is more accessible today.

It took humanity until 1973 to determine the exact number of order-5 squares–at which time Richard Schroeppel, a mathematician and computer programmer, used one hundred hours of PDP-10 computer time to arrive at this number.[3] Schroeppel prefers to divide this large number by 4 and give the total as 68,826,306 because, in addition to the eight variants obtained by rotation and reflection, there are four other variants that also preserve magical properties:

- Exchange the left and right border columns; then exchange the top and bottom border rows.

- Exchange rows 1 and 2 and rows 4 and 5, then exchange columns 1 and 2 and columns 4 and 5.

As Martin Gardner points out,[4] when these two transformations are combined with the two reflections and four rotations, the result is $2 \times 4 \times 2 \times 2 = 32$ forms that can be called isomorphic (having essentially the same structure). With this definition of isomorphic, the number of "unique" fifth-order magic squares drops to 68,826,306.

Because you can always generate a magic square from an existing magic square by subtracting each integer in the square from $N^2 + 1$, the number of "unique" fifth-order squares might be decreased further. (The new square produced by this subtraction is sometimes called the magic square's *complement.*) When the center of a 5×5 square is 13, the complement is isomorphic with the original. If it is not 13, a different square results. If we broaden the term "isomorphic" to include complements, the count of fifth-order squares drops to about 35 million.

If the notion of a complement magic square leaves you confused, consider the following fourth-order magic square on the left. Here I have exchanged all the numbers x with new numbers $N^2 - x + 1$ to produce a new magic square on the right. The two squares are said to complement one another.

1	2	16	15
13	14	4	3
12	7	9	6
8	11	5	10

Fourth-order magic square

16	15	1	2
4	3	13	14
5	10	8	11
9	6	12	7

Complement magic square

Even though 880 fourth-order magic squares might seem like a lot, the magic is quite rare when one considers that there are roughly 2,615,348,736,000 possible arrangements of consecutive integers in a 4×4 array, where no integers are used more than

once and rotations and reflections are not considered. (To compute this large number for any order N, use $N^2!/8$, where "!" is the mathematical symbol for factorial.)

As an example of a larger magic square, consider the following sixth-order square published in 1593 by Chinese mathematician Chêng Ta-wei:

27	29	2	4	13	36
9	11	20	22	31	18
32	25	7	3	21	23
14	16	34	30	12	5
28	6	15	17	26	19
1	24	33	35	8	10

Chêng Ta-wei square

Every row, column, and main diagonal adds up to 111. As preparation for the rest of the book, and for any reading you do in the magic square literature, let's review some informal definitions for various classes of diagonals using the Chêng Ta-wei square as an example:

- The *main* diagonals are the two diagonals that run from opposite corners of the square.
- The main diagonal from the top left corner cell to the bottom corner cell is called the *leading* or *left* diagonal.
- The main diagonal from the top right corner cell to the bottom corner cell is called the *right* diagonal.
- A diagonal that runs from one side of the square to the adjacent side and parallel to one of the main diagonals is called a *short diagonal*. In the Chêng Ta-wei square, one example of a short diagonal contains the numbers 32, 11, and 2. It is parallel to the right diagonal.
- Two short diagonals that contain the same number of cells and are on opposite sides of a main diagonal are called *opposite short diagonals*. The diagonal containing 9 and 29 and the diagonal containing 8 and 19 are opposite short diagonals.

- Two short diagonals that together contain N cells can form what is called a *broken diagonal.* One example is the cell containing 27 and the cells containing 24, 15, 30, 21, and 18.

I do not know if Chêng Ta-wei realized the fact that the four corners of his square add up to $\mathcal{S} \times 3/2$ or 74, as do the center four numbers and various skew-related pairs–for example, 32 + 5 + 14 + 23 (highlighted by dark squares), 25 + 12 + 16 + 21, etc. Most magic squares are like this one in the sense that their initially recognized symmetries and patterns give rise to a large number of other patterns that are not always known to the square's "author." It's as if there is a treasure chest of patterns waiting to be explored.

When consecutive integers 1 through N^2 are used, the resulting magic array of numbers is called a *pure magic square* or *traditional magic square.* In several chapters, I present unusual variations on this fundamental definition of magic square. I'll still refer to these imperfect beasts as "magic" even though they don't rigorously conform to the standard definition. To whet your appetite, consider some of the following examples. Instead of using consecutive integers starting with 1, we might use an arithmetic series starting with, say, 17 and with a difference of 3 between successive integers. A 4×4 square may be constructed[5] from the numbers 17, 20, 23, 26, 29, and so forth with magic constant $\mathcal{S} = 158$.

17	59	56	26
50	32	35	41
38	44	47	29
53	23	20	62

Imperfect magic square

Just as with traditional magic squares, a nice little formula exists to calculate the magic constant for these kinds of squares if you know the order N, the starting integer A, and the difference D between successive terms:

$$\mathcal{S} = N\left(\frac{2A + D(N^2 - 1)}{2}\right)$$

For the fourth-order magic square here, A is 17, D is 3, and N is 4. This yields $\mathcal{S} = 4 \times (34 + 45)/2 = 158$.

Imperfect magic squares can also be made from perfect magic squares simply by multiplying every number in a magic square by a constant. The results will be imperfect because the squares will not start with 1 and the numbers will not be consecutive.

There are also magic squares that are "more perfect" than perfect magic squares because they satisfy all kinds of wonderful criteria beyond the minimum of having the sums of integers in each row, column, and main diagonal equal to a constant. Various examples of these "hyper-magical" squares are discussed throughout this book.

A Brief History of Magic Squares

Lo-Shu

Some of the third- and fourth-order magic squares in the previous section were discovered centuries ago in India and China, where magic squares engraved on stones or metals were worn as charms to protect the wearer from evil and to bring good fortune. Most scholars believe the magic square originated in China and was first mentioned in a manuscript from the time of Emperor Yu, around 2200 B.C. The first square had $3 \times 3 = 9$ cells, each with Chinese characters equivalent to 1 through 9 (Figure 1).

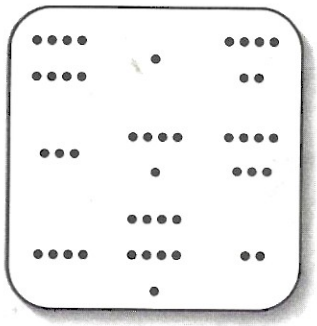
4	9	2
3	5	7
8	1	6

Yu magic square

Although historians trace references to the Yu magic square back no further than the fourth century B.C., there is a strange legend[6] that has Emperor Yu discovering the magic square while walking along the Lo River (or Yellow River), where he saw a mystical turtle crawling on the river bank. I think we can take most of

Figure 1
The first known magic square from around 2200 B.C. The Chinese characters stand for the numbers 1 through 9.

this story with a bit of skepticism, but the legend is that the turtle was ordinary in all respects except that its shell had a series of dots within squares. To Yu's amazement, each row of squares contained fifteen dots, as did the columns and diagonals. As he studied the turtle shell further, he also found that when he added any two cells directly opposite along a line through the center square, such as 2 and 8, he always got the sum of 10.



A piece of Yu's turtle shell

Emperor Yu had the turtle taken back to his palace for further study, and news of the mystical turtle began to spread to the nearby villages and eventually to other countries.

The turtle spent the rest of its easy life in Yu's court, and it became the most famous turtle in the world, having the company of famous mathematicians, kings, and international visitors. Soon the Lo-shu, as the pattern was later called, began appearing on

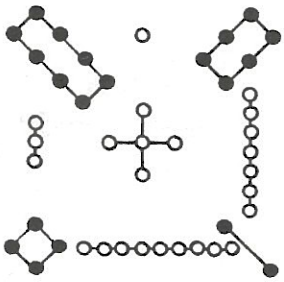


Figure 2
An ancient representation of the Lo-shu. Count the number of dots in each figure to form a 3 × 3 magic square.

charms and magic stones as it does to this day. In these charms, the arrangements of dots is usually depicted as in Figure 2, where the even numbers are represented with little filled circles and the odd numbers with open circles. (Have you ever noticed that cruise ships often feature the Lo-shu on their main deck as a pattern for the game of shuffleboard?)

From the fourth century B.C. until the tenth century, the pattern was a mystical Chinese symbol of great importance. The even numbers were identified with yin, the female principle, and the odd numbers with yang, the male principle. The central 5 represented the earth, surrounded by evenly balanced four elements in yin and yang: 4 and 9 represented metal, 2 and 7 fire, 1 and 6 water, and 3 and 8 wood.

4 (Metal)	9 (Metal)	2 (Fire)
3 (Wood)	(Earth)	7 (Fire)
8 (Wood)	1 (Water)	6 (Water)

Chinese interpretation

For centuries after their interpretation as magic squares, these squares were crucial elements of Chinese numerology, used in imperial rituals by necromancers casting spells and as the basis for prophecies and horoscopes. These kinds of squares were introduced into Europe sometime during the first millennium A.D. The first known writer on the subject was Emanuel Moschopoulus, a Greek who lived in Constantinople around A.D. 1300. He is believed to have discovered two methods for constructing magic squares.

This same square had importance in the Islamic tradition because it was believed to contain the nine letters that were revealed to Adam, that is, the first nine letters of the Arabic alphabet in the old Semitic sequence (which is used to this day when letters are taken as numbers). The even numbers in the corners are read according to their numerical value as *buduh*, and this word, sometimes interpreted as the name of a spirit, often appears on the walls to protect a building or on amulets worn around the neck or on the upper arm.

I am most fascinated by the Lo-shu square because of its ubiquity. We find it venerated by civilizations of almost every period and continent. The Mayan Indians of southern Mexico and Central America were fascinated by it, and today it is used by the Hausa people of northwestern Nigeria and southern Niger as a calculating device with mystical associations. The square was respected by the ancient Babylonians and was used as a cosmic symbol in prehistoric cave drawings in northern France.[7] In Islam, it symbolized the power of Allah's omnipresence. Members of secret societies used it as a code by drawing lines through the cells containing particular numbers in order to form secret symbols.

Geber, the eighth-century Islamic alchemist (or a later writer using his name), believed he had found the key to the elements in this simple square. He divided the square into two parts, as shown next, with the four numbers in the bottom left-hand corner adding up to 17, which was regarded as auspicious in many cultures. For Christians, 17 symbolizes the star of the Magi. For kabbalists, the seventeenth path leads to reward for the righteous. Geber noticed

that the remaining numbers of the square add up to 28, the number of letters in the Arabic alphabet.

4	9	2
3	5	7
8	1	6

Geber's square

Geber had some pretty odd ideas. One of his goals was to turn cheap metals into gold using elixirs ranging from "love-in-a-mist" to gazelle urine. By numbering various substances according to the numbers in his magic square, he thought he could determine how to turn the metal into gold. Geber also arranged the numbers around the central 5 in different ways for different magical properties. For example, the fire square and the earth square are represented as follows.

6	1	8
7	5	3
2	9	4

Fire square

2	7	6
9	5	1
4	3	8

Earth square

Magic squares such as these represented various objects in the solar system. Cornelius Agrippa (1486–1535)–physician, astrologer, and Catholic theologian–constructed squares of orders 3, 4, 5, 6, 7, 8, and 9, which he associated with the seven known (astrological) "planets": Saturn, Jupiter, Mars, the Sun, Venus, Mercury, and the Moon. Agrippa had a colorful life that included various dangerous run-ins with the Church and jobs as an occult scholar, lawyer, and military strategist. Agrippa's *De Occulta Philosophia* stimulated Renaissance study of magic and got his name into early Faust legends. Agrippa believed that a magic square containing the digit 1–which exhibits the magic constant of 1 in all directions–represented God's eternal perfection.

1

God's magic square

Agrippa and his colleagues considered the sad discovery that a 2×2 magic square could not be constructed as proof of the imperfection of the four elements: air, earth, fire, and water. Others believed that the nonexistence of a 2×2 magic square resulted from humans' original sin.

Figure 3 shows a Jupiter amulet with magic sum 34. If this configuration is engraved on a silver tablet during the time that the planet Jupiter is ruling, it is supposed to produce wealth, peace, and harmony.[8] Figure 4 shows a Mars amulet with the magic sum 65. If engraved on an iron plate or a sword when the planet Mars is in the ruling position, this amulet is said to bring success in lawsuits and victory over the owner's enemies.

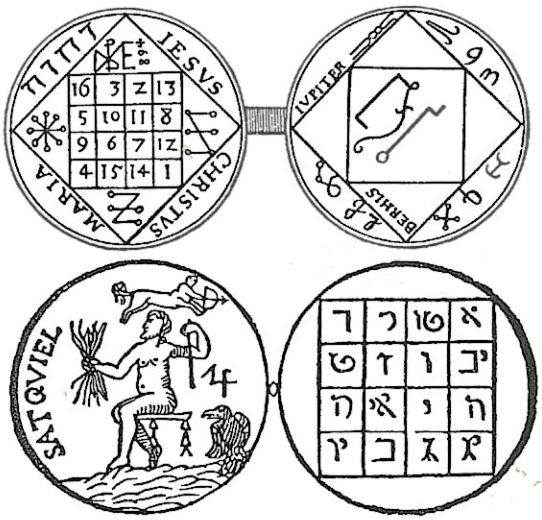


Figure 3
Jupiter amulet with magic sum 34. If this magic square is engraved on a silver tablet during the time the planet Jupiter is ruling, it is supposed to produce wealth, peace, and harmony.

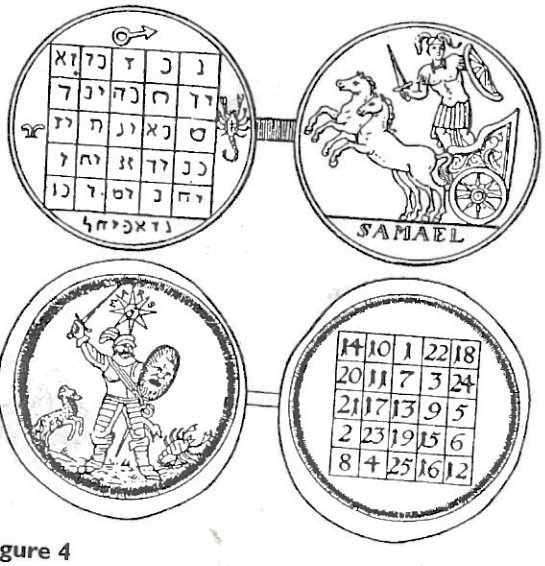


Figure 4
Mars amulet with magic sum 65.

The Arabs believed magic squares had great powers in all aspects of life. Some squares were used to protect and help lame children. The Arabs even showed certain squares to women in labor and then placed them over their wombs to facilitate births. Magic squares were also written or embroidered on the shirts of soldiers, mainly in Turkish and Indian areas. According to Annemarie Schimmel,[9] author of *The Mystery of Numbers*, people believed that such shirts had to be made by forty innocent virgins in order to work!

In Islam, the number 66 corresponds to the numerical value of the word "*Allah.*" Figure 5 is an Islamic magic square that expresses the number 66 in every direction when the letters are converted to numbers. The square's grid is formed by the letters in the word "*Allah.*" Magic squares such as this were quite common in the Islamic tradition but did not reach the West until the fifteenth century.

Quick Philosophizing

Before proceeding, let us pause and reflect upon some of these early magic squares. Why is it that Chinese emperors and empresses,

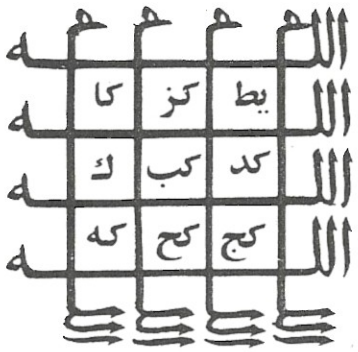


Figure 5
An Islamic magic square that expresses the number 66 in every direction. The grid is formed by the letters in the word "Allah," whose numerical value is also 66 (nineteenth century, Damascus).

Babylonian astrologer-priests, prehistoric cavemen in France, ancient Mayans of the Yucatan, and modern Hausa tribesmen were all convinced that the Lo-shu square held the secret of the universe?[10] Could they have all learned of this number arrangement from a single primal source, or is it more likely they stumbled across it independently?

The omnipresence of the Lo-shu reinforces the idea that mathematics and mysticism have fascinated humanity since the dawn of civilization. Throughout history, number arrays held certain powers that made it possible for mortals to seek help from spirits, perform witchcraft, and make prayers more potent. Numbers have been used to predict the end of the world, raise the dead, find love, and prepare for war. Even today, serious mathematicians sometimes resort to mystical or religious reasoning when trying to convey the power of mathematics.

Has humanity's long-time fascination with mathematics arisen because the universe is constructed from a mathematical fabric? In 1623, Galileo Galilei reinforced this belief by stating his credo: "Nature's great book is written in mathematical symbols." Plato's doctrine was that God is a geometer, and Sir James Jeans believed God experimented with arithmetic. Newton supposed that the planets were

originally thrown into orbit by God, but even after God decreed the law of gravitation, the planets required continual adjustments to their orbits.

What does God think of magic squares? Certainly the world, the universe, and nature can be reliably understood using mathematics. Nature *is* mathematics. The arrangement of seeds in a sunflower can be understood using Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, . . .), named after the Italian merchant Leonardo Fibonacci of Pisa. Except for the first two numbers, every number in the sequence equals the sum of the two previous. Sunflower heads, like other flowers, contain two families of interlaced spirals–one winding clockwise, the other counterclockwise. The numbers of seeds and petals are almost always Fibonacci numbers.

The shape assumed by a delicate spider web suspended from fixed points or the cross section of sails bellying in the wind is a catenary–a simple curve defined by a simple formula. Seashells, animal's horns, and the cochlea of the ear are logarithmic spirals that can be generated using a mathematical constant known as the golden ratio. Mountains and the branching patterns of blood vessels and plants are fractals, a class of shapes that exhibit similar structures at different magnifications. Einstein's $E = mc^2$ defines the fundamental relationship between energy, matter, and the speed of light. And a few simple constants–the gravitational constant, Planck's constant, and the speed of light–control the destiny of the universe. I carry this reasoning further in my book *The Loom of God,* where I state, "I do not know if God is a mathematician, but mathematics is the loom upon which God weaves the fabric of the universe."[11]

India, Germany, and Beyond

It was not until the first millennium A.D. that a fourth-order square made its appearance in society. In fact, the first record of a fourth-order square appeared in a Jaina inscription hanging over the gate at Klajuraho, India, in A.D. 1100. (Jainism is a religion and philosophy of India, founded in the sixth century B.C. by Vardhamana.) This beautiful array of numbers appears to show an advanced

knowledge of magic squares because all the broken diagonals have numbers that also sum to the magic constant 34.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Jaina magic square

For example, consider the broken diagonal $7 + 6 + 10 + 11 = 34$. These kinds of squares are also known as *diabolic*, a term discussed in chapter 2. The Jaina square contains other interesting patterns. For example, we can sum the top two rows and the bottom two rows, and the right two columns and the left two columns, revealing additional symmetries:

9	25	9	25
25	9	25	9

Row clusters

19	15	19	15
15	19	15	19

Column clusters

Today, number arrays are still used in India, drawn on paper or engraved on metal. Some squares are even regarded as being the personification of God. As pointed out by Richard Webster in *Numerology Magic*, by constructing a magic square, the individual is thought to be communicating directly with the universe's "life-force," and therefore various wishes can be granted, including the curing of piles, the alleviation of pain during childbirth, or the annoyance of an enemy. However, not all of the India magic squares are magic from a numerical point of view or contain magic sums. Consider the following square that a woman may use when searching for a husband. The numbers are drawn on a china plate with a crayon and then washed off the plate with water that the woman drinks:

24,762	24,768	24,771	25,320
24,770	24,758	24,763	25,341
24,759	24,773	24,766	25,325
24,767	24,761	24,760	25,344

Finding the perfect husband

Whenever possible, this number array should be written with a special ink known as *Ashat Gandh*. This is a mixture of several items, the most important of which is water from the Ganges River. I would be interested in hearing from readers who ascertain any significance in the use of these particular numbers.

There is even a special number array for people who feel that others are talking behind their backs. The following array, known as *Swara-e-Manafakoon*, can also be used if people have hurt you in some way. The 4×3 rectangle contains numbers of three, four, and five digits, and one of the numbers is repeated. Don't get mad, get even! If you'd like to try this square, you are supposed to recite the numbers 160 times and then keep the number array with you to prevent a recurrence of the problem.

7753	177	408	7355
7754	157	1530	17,750
17,749	177	156	17,751

Swara-e-Manafakoon

The following 3×3 square (a version of the Lo-shu) is used in India to find missing people. Written when someone vanishes without a trace, the array is hung from a tree and is thought to draw the person back home. Other 3×3 squares are used to create harmony between a man and a woman. One such square should be drawn on a Wednesday or Friday, and both people should keep a copy with them.

6	7	2
1	5	9
8	3	4

Find missing people

My favorite centuries-old European magic square is Albrecht Dürer's, which is drawn in the upper right-hand column of his etching *Melencolia I* (Figure 6). Dürer, the greatest German Renaissance artist, included a variety of small details in the etching that have confounded scholars for centuries. We seem to see the figure of a brooding genius sitting amid her uncompleted tasks. There are scattered tools, flowing sands in the glass, the magic square beneath the bell, and the swaying balance. Scholars believe that the etching shows the insufficiency of human knowledge in attaining heavenly wisdom or in penetrating the secrets of nature. As I mentioned, Renaissance astrologers linked fourth-order magic squares to Jupiter, and these squares were believed to combat melancholy (which was Saturnian in origin). Perhaps this explains the square in Dürer's engraving. Dürer's 4×4 magic square is represented as

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Dürer magic square

The square contains the first 16 numbers and has some fascinating properties. The two central numbers in the bottom row read "1514," the year Dürer made the etching. Scholars wonder if "1514" appeared accidentally or if Dürer constructed it intentionally. The rows, columns, and main diagonals sum to 34. In addition, 34 is the sum of the numbers of the corner squares (16 + 13 + 4 + 1) and of the central

Figure 6

Melencolia I, by Albrecht Dürer (1514). This figure is usually considered the most complex of Dürer's works, the various symbolic nuances confounding scholars for centuries. Why do you think he placed a magic square in the upper right? Scholars believe that the etching shows the insufficiency of human knowledge in attaining heavenly wisdom or in penetrating the secrets of nature.

2×2 square $(10 + 11 + 6 + 7)$. The sum of the remaining numbers is $68 = 2 \times 34$. Figure 7 summarizes all the amazing "34" sums from adding other configurations of cells. Just sum the numbers connected by lines. For clarity, dots represent the numbers to be added.

As you will learn in chapters 1 and 2, the Dürer square is an associated square of doubly even order because the sum of any two skew-related cells are the same. Don't worry about this termi-

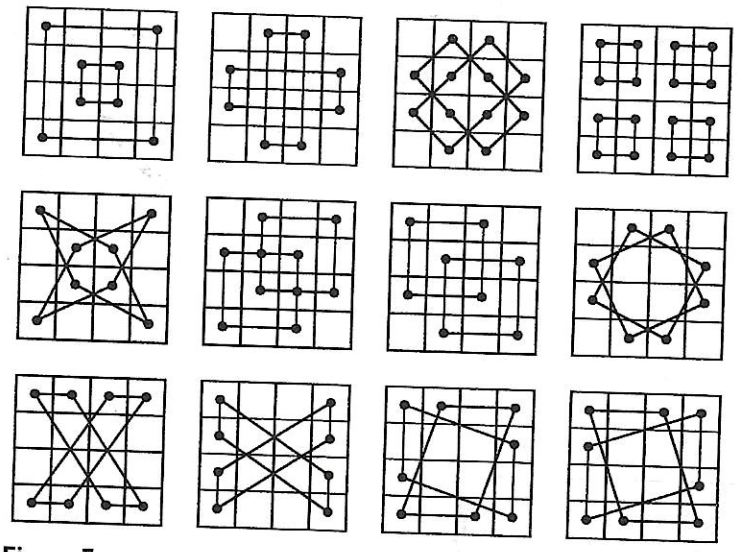


Figure 7

The Dürer square yields a magic sum in an amazing number of ways in addition to the traditional ways. The patterns, although reminiscent of the symmetries of certain crystals, tell us a good deal about the properties of this square. Try adding any of these cell configurations, where connected dots represent numbers to be added.

nology right now; the fun has just begun and doesn't require sophisticated words. For example, the sum of the squares of the integers in the first and second rows equals the sum of the squares of the integers in the second and fourth rows, equals the sum of the squares of the integers in the first and third rows, equals the sum of the squares of the integers in the third and fourth rows, equals the sum of the squares of the integers not in the main diagonals, equals 768! All of this is also true for the columns. The Dürer square contains other interesting patterns. For example, like the Jaina square, we can sum the top two rows and the bottom two rows, and the right two columns and the left two columns, revealing additional symmetries:

21	13	13	21
13	21	21	13

Row clusters

19	15	15	19
15	19	19	15

Column clusters

Even better, the sum of the cubes of the integers in the main diagonals equals the sum of the cubes of the integers not in the main diagonal.

256	9	4	169
25	100	121	64
81	36	49	144
16	225	196	1

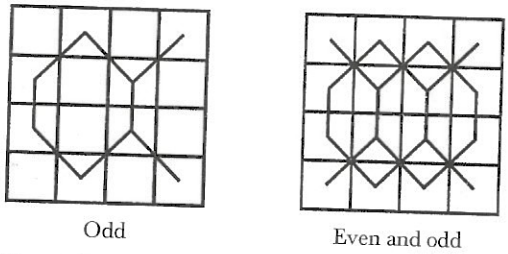
Dürer squared

4096	27	89	2197
125	1000	1331	512
729	216	343	1728
64	3375	2744	1

Dürer cubed

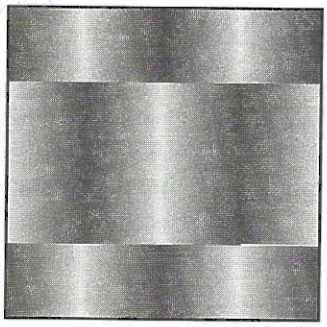
I wonder if Dürer thought there was something profound about the magic sum 34? Probably not, but don't you wonder why it is even possible to create such a square with all the marvelous symmetries? Additional features of the Dürer square are discussed in chapter 3. For now, I will whet your appetite by pointing out yet another symmetry, simply by connecting the odd and even numbers (Figure 8).

It is also possible to use bicolored tiles to highlight additional symmetry and beauty. For example, I have used a light-to-dark tile to represent odd numbers and a dark-to-light tile to represent even squares:



Odd Even and odd

Figure 8
Connecting odd and even numbers in the Dürer square.



Dürer magic tile

Try this coloring method on other magic squares of your own design.

Templar Magic Square

The templar magic square is a particular arrangement of the words in the Latin sentence "*Sator Arepo tenet opera rotas*," which has been translated variously as "The farmer Arepo keeps the world rolling," "Farmer Arepo keeps the wheels at work," "Farmer Arepo steers the plough," "The sower Arepo works with the help of a wheel," or "Arepo the farmer holds the works in motion." Arranged both vertically and horizontally, the seemingly unimportant word TENET forms the two arms of a central cross. This first-century square has been found in excavations of ancient Pompeii, Italy. The square continued to be used during the nineteenth century in Europe and the United States for protection against fire, sickness, and other disasters.

S	A	T	O	R
A	R	E	P	O
T	E	N	E	T
O	P	E	R	A
R	O	T	A	S

Templar magic square

In Rome during the Middle Ages this square was inscribed on a variety of common, everyday objects, such as utensils and drinking vessels. It was also placed above doorways. People believed that the square had magical properties and that making it visible would ward off evil spirits. I am not sure why there are conflicting translations of the Latin. For example, some have translated the words in this square less literally, as in "The Creator (or Savior) holds the working of the spheres in his hands," or "The sower [i.e., God] in his field controls the workings of his tools [i.e., us]." Perhaps it is these more expansive interpretations that have given the square meaning through the ages.

Most authors who have reported on this square in the past have not noticed that the following hidden message arises if the square's contents are rearranged:

```
          P
          A
    A     T     O
          E
          R
P A T E R N O S T E R
          O
          S
          T
    O     E     A
          R
```

This translates to "Our Father, Our Father" in the shape of a cross, which must have been a potent Christian message posted near doorways to let Christians in the neighborhood know that sympathetic families were nearby. The four remaining letters may represent alpha and omega–the beginning and the end. Note that this hidden message

is an example of what codebreakers call steganography, an encryption technique that conceals a message so that the casual viewer doesn't even know there is a secret message waiting to be found.

The French Connection

Humanity did not know very much about magic squares until the seventeenth century, when there was an explosion of interest in France by mathematicians such as Simon de la Loubère, Claude Gaspar Bachet de Méziriac, Philippe de la Hire, and others discussed in chapter 1. In 1838, Jean-Yves Violle published three volumes on magic squares that contained hundreds of illustrations. It was during this period that the 880 different fourth-order magic squares were found, not counting rotations and reflections.

Although research in magic squares still continues unabated today, some researchers in the early 1900s seemed to think that there was not much more to know about these impressive number patterns. For example, world-renowned puzzlist Henry E. Dudeney (1857–1930) wrote in his 1917 book *Amusements in Mathematics*,

> Of recent years many ingenious methods have been devised for the construction of magics (magic squares), and the law of their formation is so well understood that all the ancient mystery has evaporated and there is no longer any difficulty in making squares of any dimensions. Almost the last word has been said on this subject.[12]

Dudeney's assertions proved to be not true. In fact, he quickly invalidated his own prediction when he continued to write about magic squares. However, Dudeney did not think there was much value in classifying magic squares. Dudeney wrote, "A man once said that he divides the human race into two great classes: those that take snuff and those who do not. I am not sure that some of our classifications of magic squares are not almost as valueless."[13] Coincidentally, one of the world's most important, thorough, and beautiful books on magic squares, W. S. Andrews's *Magic Squares*

and Cubes, was also published in 1917, at about the same time Dudeney was making these sorts of statements.

To my mind, we have certainly made remarkable progress, and the field of magic square study is wide open. Consider that *Homo sapiens* took about 20,000 years to discover the Yu magic square (starting from our origins as anatomically modern *Homo sapiens* in the Upper Paleolithic period). It took another 3500 years for magic squares to permeate Europe, thanks in part to Emanuel Moschopoulus. About 390 years after that, Bernard Frénicle de Bessy published the 880 different fourth-order magic squares. Then, 280 years later, Richard Schroeppel determined all order-5 squares. About 20 years later, John Hendricks discovered four-, five-, and six-dimensional magic hypercubes. If this isn't progress, nothing is! What do you think comes next?

Geometrical Diagrams

Throughout this book I have drawn *geometrical diagrams* that reveal some of the startling symmetries of magic squares not obvious from the number arrays. The traditional method for drawing such diagrams, which probably dates back more than a century,[14] lists all of the numbers in the square and connects those numbers that are in the same rows. Usually the geometrical diagrams are presented in two columns of interconnected dots, and you'll see some of my favorite diagrams in Gallery 1, which includes information on the Franklin square. As I mention in Gallery 1, I like to meditate upon the geometrical diagrams as if they are mandalas–Hindu and Buddhist symbolic diagrams used in the performance of sacred rites and as an instrument of meditation. Traditionally, mandalas were painted on paper or cloth, drawn on a carefully prepared ground with white and colored threads or with rice powders, fashioned in bronze, or built in stone. Perhaps mandalas of the twenty-first century will include the geometric diagrams for huge magic squares with dimensions greater than 1000×1000.

The diagrams are not just beautiful to look at but also have a practical side because they give us hints about the existence of higher-

dimensional magic figures. They also suggest regular laws in the arrangement of squares that persist from low- to high-order squares.

To whet your appetite, here is an 8×8 magic square,[15] with magic constant 260, and its geometrical diagram (Figure 9). (Incidentally, notice that skew-related cells add up to the same sum, for example, $29 + 36 = 8 + 57 = 33 + 32 = 65$.) To help you find your way, take a look at the first row in the magic square (1, 7, 59, 60, 61, 62, 2, 8) and see how it is represented by the line that intersects these numbers in the upper left part of the geometrical diagram.

1	7	59	60	61	62	2	8
16	10	54	53	52	51	15	9
48	47	19	21	20	22	42	41
33	34	30	28	29	27	39	40
25	26	38	36	37	35	31	32
24	23	43	45	44	46	18	17
56	50	14	13	12	11	55	49
57	63	3	4	5	6	51	64

Eighth-order magic square

The geometrical diagram makes it clear that there are two different underlying symmetries that characterize the order-8 square. Gallery 1 shows an order-8 square with *three* underlying symmetries.

The best part about these diagrams is that the structures do not seem random. Symmetrical themes are always repeated, like movements in a symphony or atoms in a crystal.

I would be interested in hearing from readers who have invented other kinds of geometrical diagrams. For example, the diagrams might be modified to form circular geometrical diagrams to reveal hidden structure and order. This approach is similar to the traditional two-column interconnected dots approach that connects numbers in the same magic square row, but instead you

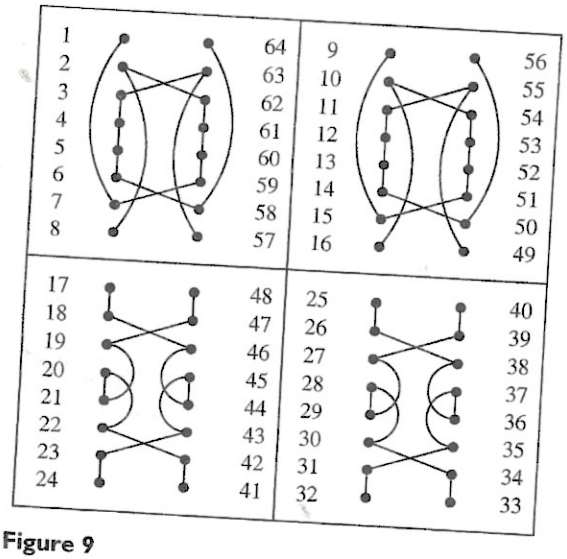


Figure 9
Geometrical diagram for an 8 × 8 magic square.

can abolish the dot columns in favor of a more general diagram that positions consecutive integers around the circumference of a circle. These circular (or other kinds of) geometrical diagrams might be quite convenient because they fit on one page, even for huge magic squares. The traditional geometrical diagram can also be made to fit on a single page by chopping it into subsections.

Zen Mathematics

The Gohonzon of the Soka Gakkai

At that time in the Buddha's presence, there was a tower adorned with the 7 treasures, 500 yojanas in height and 250 yojanas in width and depth. . . . It had 5000 railings, 10,000 rooms, . . . 10,000,000,000 jeweled bells.

–The Lotus Sutra

An elephant is contained in a blade of grass.

–Ancient Buddhist saying

In previous sections, I've discussed how magic squares were used in magic, divination, alchemy, and other mysterious settings. The use of mathematical and geometrical patterns for magic and meditation has a rich history. In Japan, there exists a religious group, the members of which meditate using visual aids resembling fractals and other ornate geometries. *Soka Gakkai* is an association of laypeople who practice Nichiren-sho-shu Buddhism. The association was founded in Japan in 1937; however, the practice of Nichiren-sho-shu Buddhism dates back centuries to the followers of the Japanese Buddhist saint Nichiren (1222–1282), an ardent nationalist who attacked the established contemporary religious and political institutions of Japan. After exploring different facets of Buddhist scripture, Nichiren concluded that only one scripture gave the truth–the *Lotus Sutra* (the scripture of the Lotus of the Good Law).

One way the Soka Gakkai pay devotion to the Lotus Sutra is through a ritual drawing, or mandala, called the *gohonzon.* The gohonzon contains the names of divinities mentioned in the sutra arranged around the name of the Lotus Sutra. In particular, the gohonzon contains the sacred phrase *Nam-myoho Renge-kyo* (Devotion to the Lotus Sutra) written vertically in the center, around which are arranged names of various deities–perhaps to safeguard the seven-ideograph phrase. Today, millions of followers chant the phrase thousands of times a day.

The gohonzon mandala, an example of which is shown in Figure 10, is placed on an altar in Nichiren temples, as well as in the homes of Nichiren Buddhists. The original *dai gohonzon*–the supergohonzon for all humanity–resides in a temple in Kyoto. No photographs of the dai gohonzon are allowed, and no pictures have been published. To the Nichiren Buddhists, the gohonzon symbolizes the superiority of the Lotus Sutra over other religions and sects. Nichiren himself drew many gohonzon mandalas that he gave to his followers. Since the followers were only entitled to the mandalas while they lived, the mandalas were returned to the temples after the deaths of the followers.

The gohonzon reminds me of a fractal; all the different renditions I have seen have visual details on at least two different size scales. When I talked to Gary Adamson, expert and practitioner of Nichiren-sho-shu Buddhism, he indicated that the gohonzon mandala represents the

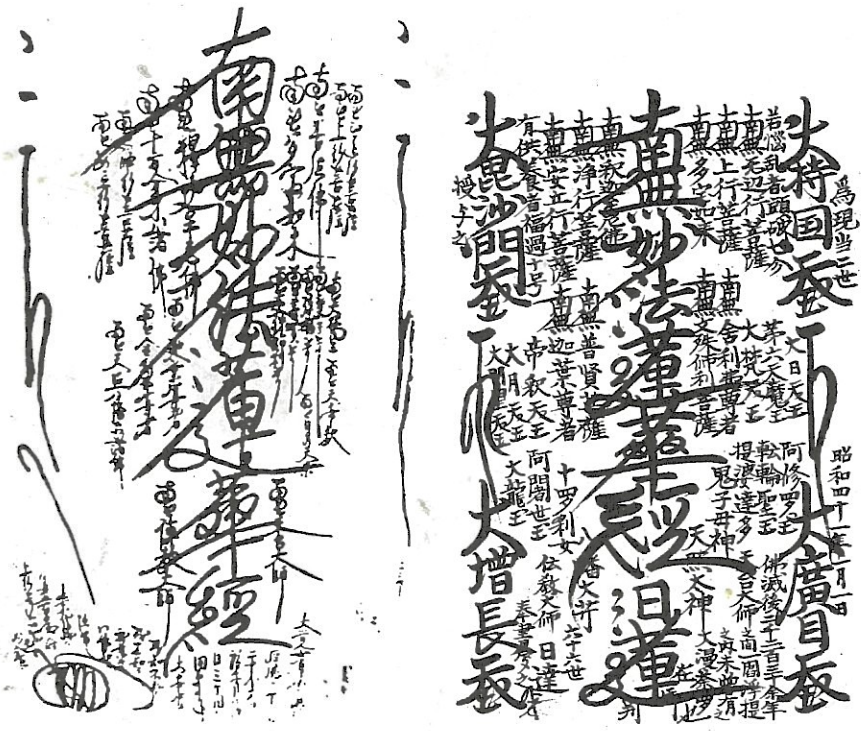


Figure 10
Two gohonzons of Nichiren (thirteenth century).

Cosmos, the totality of existence, or, in modern mathematical terms, "the class of all sets." Further, he believes that the gohonzon's basic theme is "the whole is greater than the sum of its parts." Gary says, "When people chant to the gohonzon, they are not looking at individual parts of the fractal figure, but the complete whole." Gary sent me the ancient gohonzons in Figure 10 from the Nichiren Shoshu sect. The world of Buddahood resides at the top of the figure, and hell is below. Buddhism has a classification of ten dimensions, the topmost dimension being that of enlightened Buddhas.

Nichiren Buddhism contains various numerical curiosities–just as do many mystical groups or religions mentioned in this book in relation to magic squares. For example, one of the main elements of Nichiren's teachings is the theory of *ichinen sanzen* (1 thought, 3000) which means that the mind, at any moment, contains 3000 different

aspects of the universe. To arrive at the figure 3000, Buddhists consider the human mind to be in one of ten states at any particular moment: hell, craving, animalness, anger, tranquility, rapture, intellectual pleasure, truth, mercy and wisdom, and, finally, Buddhahood. In each state of mind there are the same ten aspects, bringing the total number of states of mind to 100. There are an additional ten factors of existence–appearance, nature, entity, power, action, cause, relationship, effect, reward, and consistency–which bring the 100 states to 1000. Finally, the 1000 is multiplied by 3 to take account of the three categories of human life: physical and mental abilities, the relationship between the individual and other members of his or her community, and the environment in which an individual personality exists.

Makiguchi Tsunesaburo, the founder of Nichiren-sho-shu Buddhism, suggests that the goals of life are beauty, gain, and goodness. Through Nichiren Buddhism, an individual may attain these goals.

Gary Adamson indicates that the gohonzon is unlike other popular visual meditation aids, such as magic squares or the shri-yantra of India, because the gohonzon is the most free-flowing of all mandalas, having a random component along with a beautiful structure "analogous to many natural phenomena like twigs of a tree." The shri-yantra, an ancient Vedic symbol that represents the seed of the universe–a place that exists beyond space and time–is purely geometric, using triangles, squares, circles, and parallelograms (Figure 11). Some medieval and later Hindu temples in India contain shrines with shri-yantra engravings. The shri-yantra is also engraved on foil and placed in a metallic case worn as an amulet for health.

The shri-yantra is ripe for magic-sum exploration. For example, can you put consecutive numbers at each corner and intersection of the triangles so that the numbers along each line segment produce the same magic sum? So far no one has been able to create such a labeling of the points on the figure.

The mouse child uses the image as a yantra to meditate on nothing and infinity.

–*Daedalus,* 1980

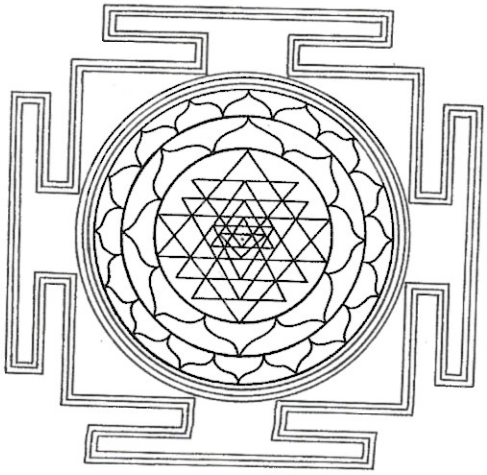


Figure 11
The shri-yantra of India, a geometrical diagram used as an aid to meditation. This yantra is composed of nine juxtaposed triangles arranged to produce forty-three small triangles. Four of the nine primary triangles point upward, representing male cosmic energy, and five point downward, symbolizing female cosmic power. In southern India, the shri-yantra is an object of worship.

Hermetic Geometry

We who are heirs to three recent centuries of scientific development can hardly imagine a state of mind in which many mathematical objects were regarded as symbols of spiritual truths.

–Philip J. Davis and Reuben Hersh, *The Mathematical Experience*

Completely involved as he was in Hermetism, Bruno could not conceive of a philosophy of nature, of number, of geometry, of a diagram, without infusing into these divine meanings. Bruno based memory on celestial images which are shadows of ideas in the soul of the world, and thus unified the innumerable individuals in the world and all the contents of memory.

–Frances Yates, *Giordano Bruno and the Hermetic Tradition*

Through the ages, various patterns, such as in magic squares or in the gohonzon, have been created to induce spiritual forces to influence material forces. Perhaps the most geometrical of these mystical figures comes from Hermetic[16] geometry, where the diagrams represent pure celestial forms. The design on paper was supposed to induce a resonance with its celestial counterpart, and, as a result, the figure was thought to have various powers. For example, the symmetrical patterns were used to achieve

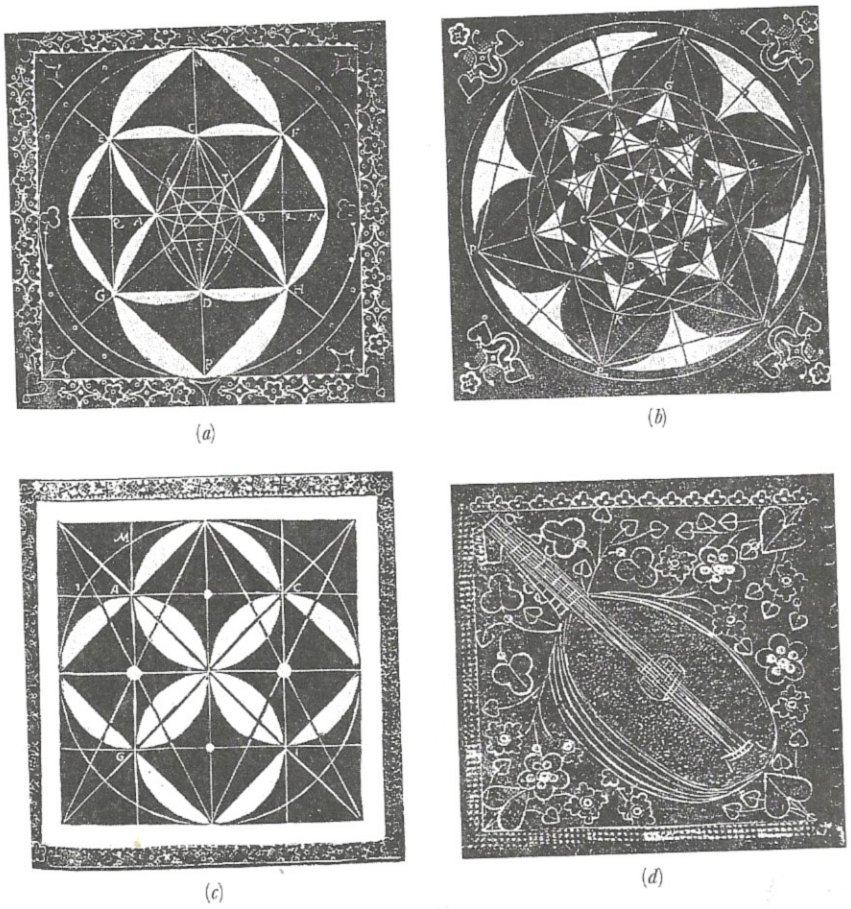


Figure 12
The Hermetic figures of Giordano Bruno (1588): (a) Figura Mentis, (b) Figura Intellectus, (c) Figura Amoris, (d) Zoemetra.

personal gain, cure diseases, find love, or harm one's enemies. Figure 12 shows Hermetic designs from 1588 by Giordano Bruno. The Hermetic geometers thought that these designs were keys to the universe.

Giordano Bruno (1548–1600) was an Italian philosopher, mathematician, astronomer, and occultist. Some of his theories anticipated modern science. For example, he rejected the traditional Earth-centered universe and believed in a multiplicity of worlds. He also believed that the Bible should be followed for its moral teaching but not for its astronomical implications.

Giordano Bruno's patterns are of particular interest because of their geometrical and (often) recursive shapes. Occasionally, Bruno adds objects such as serpents and lutes. If you look closely, you can see that Figure 12*c* ("Figura Amoris") actually has the word "Magic" written in the diagram.

Bruno had a dramatic life. He thought himself a messiah, which was not uncommon for magicians of the Renaissance. The Venetian Inquisition was not pleased with Bruno's ideas and forced him to justify them. He gave the Inquisition a detailed technical account of his philosophy as if he were lecturing great scholars. He told them that he believed the universe to be infinite, because the infinite divine power would not produce a finite world. The Earth was a star, as Pythagoras thought, like the moon, other planets, and other worlds that were infinite in number.

At the end of the Venetian trial, Bruno fully recanted various heresies of which he was accused and threw himself at the mercy of the judges. By law, his case was sent on to Rome, where he was imprisoned as the trial dragged on for years. Finally, in 1599, a judge listed eight heresies taken from the works of Bruno that Bruno was required to repudiate. Bruno countered that there was nothing in his works that was heretical and that the church was merely misinterpreting them. Bruno finally declared that he did not even know what he was expected to retract. As a result, Bruno was sentenced as an impenitent heretic. His death sentence was read, and Bruno addressed his judges, "Perhaps your fear in passing judgment on me is greater than mine in receiving it." He was gagged and burned alive in Rome on February 17, 1600.

Unfortunately, today we do not have the report of the Venetian Inquisition, and so we do not know the eight heretical propositions that Bruno was required to recant. Scholars believe they have something to do with God's infinity implying an infinite universe, the mode of creation of the human soul, the motion of the Earth, the soul, the multiplicity of worlds, and the desirability of using magic. Other problems arose from the fact that Bruno believed that Moses performed his miracles by magic and that Christ was a magician.

Frances Yates, writing in *Giordano Bruno and the Hermetic Tradition*, notes

> The Renaissance magic was turning towards number as a possible key to operations, and the subsequent history of man's achievements in applied science has shown that number is indeed a master-key, or one of the master-keys, to operations by which the forces of the cosmos are made to work in man's service. However, neither Pythagorean number, organically wedded to symbolism and mysticism, nor Kabalistic conjuring with numbers in relation to the mystical powers of the Hebrew alphabet, will of themselves lead to the mathematics which really work in applied sciences.[17]

Other, more recent mystics have also used geometrical diagrams in spiritual realms. For example, Figure 13 shows the Sahasra chakra, or energy centers, of Tantric literature. This diagram was drawn in the mid-1800s by Yogi Lahiri Mahasaya *(1828–1895)*.

I can easily imagine future generations, or even alien civilizations, that place great faith in magic squares, like some of our ancients did. In our modern era, God and mathematics are usually placed in totally separate arenas of human thought. But this has not always been the case, and even today many mathematicians find the exploration of mathematics akin to a spiritual journey. The line between religion and mathematics becomes indistinct. In the past, the intertwining of religion and mathematics has produced useful results and spurred new areas of scientific thought. Consider, as just one small example, numerical calendar systems first developed to keep track of religious rituals. Mathematics, in turn, has affected religion because mathematical reasoning and proofs have contributed to the development of theology.

In many ways, the mathematical quest to understand infinity and higher-dimensional magic squares parallels mystical attempts to understand God. Both religion and mathematics attempt to express relationships between humans, the universe, and infinity. Both have arcane symbols and rituals and impenetrable language. Both exercise the deep recesses of our minds and stimulate our imagination. Mathematicians, like priests, seek ideal, immutable, nonmaterial truths and then often try to apply these truths in the real world.

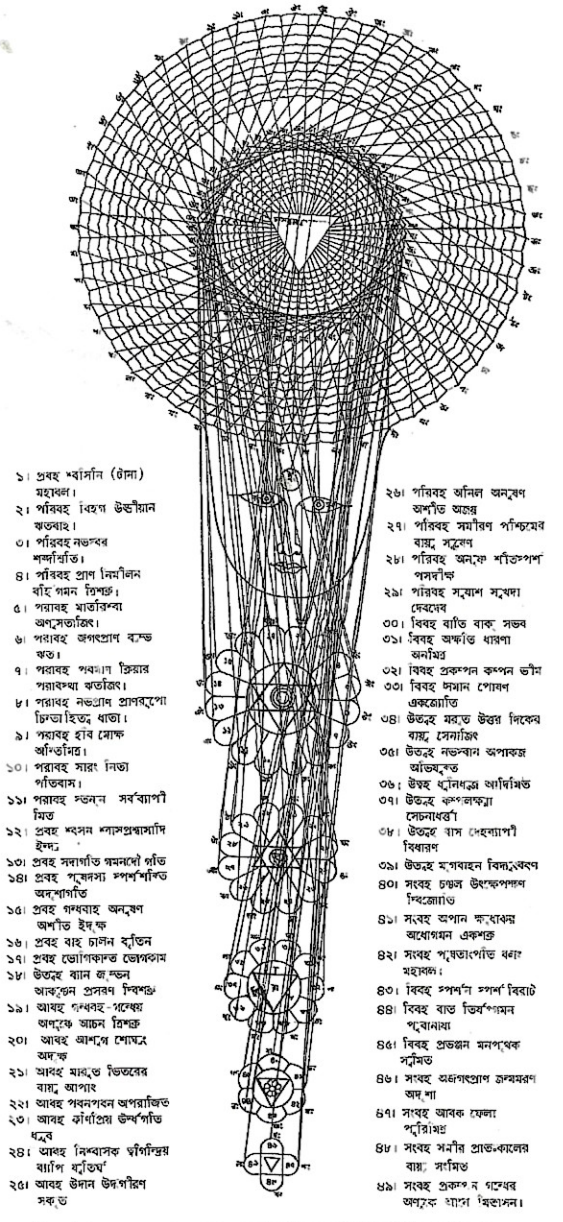


Figure 13

The Sahasra chakra, or energy centers, of Tantric literature. The diagram is used to understand how forty-nine types of breath are associated with the six centers. The thousand petals' center at the head is described as the "door of the Guru"—utter tranquility through inhaling and exhaling. Visually compare the nested geometrical objects with the Hermetic figures of Giordano Bruno. (Drawing by Yogi Lahiri Mahasaya [1828–1895] communicated to Swami Satyeswarananda Giri to Gary Adamson to me.)

CHAPTER ONE

Magic Construction

We are in the position of a little child entering a huge library whose walls are covered to the ceiling with books in many different tongues. . . . The child does not understand the languages in which they are written. He notes a definite plan in the arrangement of books, a mysterious order which he does not comprehend, but only dimly suspects.

–Albert Einstein, *The Saturday Evening Post*

Categories

When you gaze at magic squares with their amazing properties and hidden symmetries, it's sometimes difficult to believe that there are easy-to-remember ways to construct many of them using simple rules. In fact, handbooks of "mental magic" often give these methods as "secret" ways to impress audiences.[1] The mentalist shows a large, empty square to the enthralled onlookers and, with a flourish and a cry of "*voilà!*", creates a magic square with astonishing ease. Imagine yourself being able to write down a large magic square in under a minute in front of an adoring audience. In this chapter, I'll introduce you to a few methods for constructing some of the traditional magic squares. If you are interested in pursuing the construction of some of the more elaborate magic squares, several books[2] in the For Further Reading section provide additional material and food for thought.

Various methods for constructing magic squares have evolved through the ages. When considering these methods, it is useful to categorize magic squares in three classes:

- Magic squares of *odd order* (squares where the order N is of the form $2m+1$, where m may be any positive integer 1, 2, 3, etc.).
- Magic squares of a *doubly even order* (where the order N is of the form $4m$, such as 4, 8, 12, 16, 32, etc.). The order of a doubly even square can be divided by 2 and 4.
- Magic squares of a *singly even order* (where N is of the form $2(2m+1)$, such as 2, 6, 10, 14, 18, 22, etc.). The order of a singly even square can be divided by 2 but not 4.

Generation methods for these kinds of squares vary in complexity, and for brevity I focus mostly on magic squares of odd order. Corresponding general methods for the construction of even-order magic squares are difficult to implement with pencil and paper, and magic squares of singly even orders are generally the most difficult of all to construct.

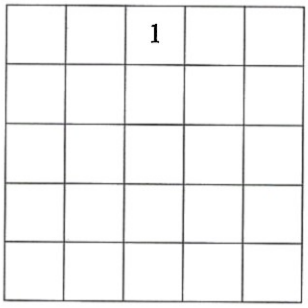
De la Loubère's Method

The smallest possible square of an odd order greater than 1 is the third-order square from the Introduction that has only one arrangement if we discount reflections and rotations:

4	9	2
3	5	7
8	1	6

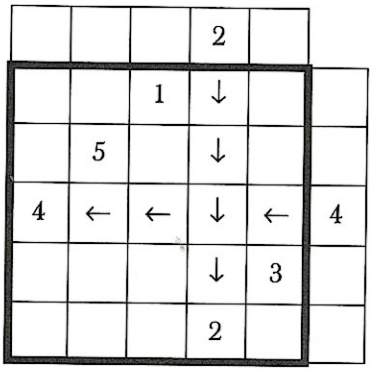
Third-order magic square

In 1693, mathematician Simon de la Loubère suggested a method to create any odd-order magic square. He learned of it while ambassador to Siam from the court of Louis XIV. You can think of this as the "upward-right method" for reasons that will become clear. Start by placing a 1 in the central upper cell. In this example, we will construct a fifth-order magic square.



De la Loubère's first step

Proceed diagonally upward to the right and place a 2 in an imaginary square outside the table. To visually represent this process, let's temporarily add another row and column to the square. Because the 2 is outside the square, we bring it to the bottom of the column. Next, write a 3 upward to the right of the 2.



De la Loubère's movements

Next, write a 4 upward and to the right of the 3. Because 4 falls outside the original square, place it at the opposite end of the row, and write a 5 upward to the right. We cannot write a 6 upward and to the right of the 5 because the cell is occupied. Therefore, we write the 6 below the 5. We proceed until the 10 falls outside of the original square and continue the process. Notice that 16 falls outside the corner square and is written beneath the 15, as is the case when encountering an occupied square.

	18	25	2	9	
17	24	1	8	15	17
23	5	7	14	16	23
4	6	13	20	22	4
10	12	19	21	3	10
11	18	25	2	9	

De la Loubère's method completed

In summary

1. Start by placing 1 in the center cell, top row.
2. Whenever a top edge is reached, the integers are continued diagonally upward to the right at the bottom of the next column.
3. When the right-hand edge is reached, the integers are continued at the last cell to the left of the next highest row.
4. When a cell is reached that is already filled, drop down one cell and continue.
5. When the upper right corner cell is reached, drop down one row.

By rotations and reflections, seven other magic squares can be constructed from this method. Once you have memorized this simple approach, you can amaze your friends by generating odd-order squares of higher order. Additionally, you can start with a 1 in *any* cell and always generate a square that is magic in rows and columns but not necessarily diagonals.

You can also use de la Loubère's method to form imperfect magic squares that start with numbers other than 1. For example, the following square was created by starting with 3, and each succeeding integer was obtained by adding 2 to the preceding integer:

17	3	13
7	11	15
9	19	5

De la Loubère's square

Try this approach with other starting numbers and with other differences between succeeding integers.

It's also possible to use de la Loubère's method to form imperfect magic squares with more complicated number sequences. Let us define the horizontal difference ☞ as the difference between each succeeding integer in each group of *N* integers (represented as rows in the following squares). The vertical difference ☝ is defined as the difference between the last integer of the first *N* integers and the first integer of the next set of *N* integers. ☞ and ☝ do not have to be equal but must remain constant throughout. So you can more easily get a feel for these parameters, here are some examples of ☞ and ☝ in number arrays. (These are not magic squares.)

1	2	3	4
5	6	7	8
9	10	11	12
13	14	16	16

☞ = 1, ☝ = 1

1	2	3	4
6	7	8	9
11	12	13	14
16	17	18	19

☞ = 1, ☝ = 2

2	4	6	8
9	11	13	15
16	18	20	22
23	25	27	29

☞ = 2, ☝1

Here is a magic square created by de la Loubère's method using a starting number of 3, where ☞ = 2 and ☝ = 3:

19	3	14
7	12	17
10	21	5

Magic square, ☞ = 2, ☝ = 3

In other words, the groups 3, 5, 7; 10, 12, 14; and 17, 19, 21 were used to create this square.

When a magic square of order N is constructed in these ways, it is possible to predict in advance the magic constant $\mathcal{S}$ for the square using the formula

$$\mathcal{S} = \tfrac{1}{2}(N^3 + N) + N(\boxdot - 1) + (\kappa - N)[N(☞ - 1) + (☝ - 1)]$$

where $\mathcal{S}$ is the magic constant, N is the order of the square, $\boxdot$ is the number in the starting cell, κ is the sum of the numbers $1 + 2 + 3 + \cdots + N$, ☞ is the horizontal difference, and ☝ is the vertical difference. For the previous magic square, we get $\mathcal{S} = \tfrac{1}{2}(3^3 + 3) + 3(3 - 1) + (6 - 3)[3(2 - 1) + (3 - 1)] = 15 + 6 + 3 \times (3 + 2) = 36$. Notice that for a pure magic square, $\boxdot = ☝ = ☞ = 1$, and the formula simplifies to the one provided in this book's introduction.

The formula becomes much simpler if you create a magic square by starting with a number other than 1 and filling all cells with consecutive numbers. If we denote the number in the starting cell by $\boxdot$ and the order by N, the magic constant of this kind of square can be determined by

$$\mathcal{S} = \tfrac{1}{2}(N^3 + N) + N(\boxdot - 1)$$

Claude Gaspar Bachet de Méziriac's Method

Claude Gaspar Bachet de Méziriac (1581–1638) used a method similar to de la Loubère's to generate magic squares. Bachet joined the Jesuit Order in 1601 but left the order the following year after an illness. For most of his life Bachet lived in comparative leisure on his estate at Bourg-en-Bresse. He wrote books on mathematical puzzles and tricks that formed the basis of many later books on mathematical recreations.

To use Bachet's method, start with a 1 in the cell just above the center cell. Follow the diagonal rule of de la Loubère's method until an occupied cell is reached.

23	6	19	2	15
10	18	1	14	22
17	5	13	21	9
4	12	25	8	16
11	24	7	20	3

Bachet's method

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

De la Loubère's method

When this condition arises, as with the 5 in this example, the integers are continued in the same column *two cells higher.* If this is not possible because it would be above the magic square, as with 10, move down to the bottom cell in that column instead. When the upper right corner cell is reached, as with 15, the sequence is continued in the cell just above the right bottom corner cell. A comparison of Bachet's method and de la Loubère's method is shown here for a fifth-order square.

The following 15×15 square was constructed in the early 1600s.[3] It was reported in the *Arithmetischer Cubic-cossisher Lustgarten* ("Arithmetic-Algebraic Pleasure Garden," Tübingen, 1604) by the German mathematician and Rosicrucian Johann Faulhaber (1580–1635). (The Rosicrucians are a brotherhood combining elements of mystical beliefs with an optimism about the ability of science to improve the human condition.) The square's author is unknown, and I include it here because it provides an instructive example of a construction method for odd-order magic squares that is similar to Bachet's method.

Start with a 1 in the cell to the right of the center cell, and continue diagonally upward to the right. A diagonal that reaches the right-hand border column of the square is continued at the left-hand side of the square. Similarly, when a diagonal reaches the top row of the square, it is continued in the bottom row. These two rules are illustrated by the jump between 7 and 8 and by the jump between 8 and 9. When a diagonal encounters a cell that is already occupied, it restarts at the cell two steps to the right in the same row. This is shown by the jump between 15 and 16, where the 1 cell is occupied. An interesting case occurs at cell 120 at the upper

8	121	24	137	40	153	56	169	72	185	88	201	104	217	120
135	23	136	39	152	55	168	71	184	87	200	103	216	119	7
22	150	38	151	54	167	70	183	86	199	102	215	118	6	134
149	37	165	53	166	69	182	85	198	101	214	117	5	133	21
36	164	52	180	68	181	84	197	100	213	116	4	132	20	148
163	51	179	67	195	83	196	99	212	115	3	131	19	147	35
50	178	66	194	82	210	98	211	114	2	130	18	146	34	162
177	65	193	81	209	97	225	113	1	129	17	145	33	161	49
64	192	80	208	96	224	112	15	128	16	144	32	160	48	176
191	79	207	95	223	111	14	127	30	143	31	159	47	175	63
78	206	94	222	110	13	126	29	142	45	158	46	174	62	190
205	93	221	109	12	125	28	141	44	157	60	173	61	189	77
92	220	108	11	124	27	140	43	156	59	172	75	188	76	204
219	107	10	123	26	139	42	155	58	171	74	187	90	203	91
106	9	122	25	138	41	154	57	170	73	186	89	202	105	218

Johann Faulhaber's fifteenth-order magic square

right, where the progress of the diagonal is blocked by cell 106 at the lower left. Because 120 is the last cell in the top row, the two-step shift of the diagonal has to be made at the left end of the top row, and 121 is entered in the second cell from the left. The diagonal is then continued from cell 122 in the bottom row.

The procedure is continued until the last number, 255, is entered two steps away from the starting cell with 1. The sum of *all* the numbers in any fifteenth-order magic square is $N^2(N^2 + 1)/2 = 225 \times 226/2 = 25{,}425$. The sum for every row, column, and main diagonal is $25{,}425/15 = 1695$.

The eclectic Johann Faulhaber was fascinated by magic squares even though he was trained as a weaver. Later, he was taught mathematics in Ulm, Germany, and showed such promise that the city of Ulm appointed him city mathematician and surveyor. He opened his own school in Ulm and was in great demand because of his skill in fortification work. His expertise enabled him to work on fortifications in Basel, Frankfurt, and many other cities. He also designed waterwheels in Ulm and made mathematical and surveying instruments, particularly ones with military applications.

Faulhaber was an algebraist and important for his work explaining logarithms and sums of powers of integers. Faulhaber collaborated with many eminent people of his day, such as astronomer Johannes Kepler and Ludolph van Ceulen, the German mathematician who calculated pi to thirty-five places. His scientific and Rosicrucian beliefs made a major impression on philosopher René Descartes.

Philippe de la Hire's Method

French mathematician Philippe de la Hire (1640–1719) translated Emanuel Moschopoulos's essays on magic squares and collected many magic square theorems. De la Hire's method of creating magic squares of singly even order makes use of two generating squares that get summed together.

6	2	3	4	5	1
1	5	3	4	2	6
1	2	4	3	5	6
6	2	4	3	5	1
1	5	4	3	2	6
6	*5*	*3*	*4*	*2*	*1*

A

0	30	0	30	30	0
6	6	24	24	6	24
18	12	12	12	18	18
12	18	18	18	12	12
24	24	6	6	24	6
30	*0*	*30*	*0*	*0*	*30*

B

6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	20	22	21	17	13
24	29	10	9	26	12
36	*5*	*33*	*4*	*2*	*31*

C

Let's try to use his method to construct a sixth-order magic square *C* from two other 6×6 squares *A* and *B*. In square *A*, fill the cells in the main diagonals (highlighted) with the numbers 1

through N beginning at the top of one diagonal and at the bottom of the other diagonal. In our sixth-order example, $N = 6$, and we fill the diagonals with numbers 1 through 6. In the next step, you have lots of leeway. Fill each of the remaining cells of the first column with either N or its "complement" (the integer 1) in any way you like, providing that there is the same number of each of these integers in the column. (Here we use the term "complement" of number x to mean $N - x + 1$, although authors also sometimes use "complement" when referring to $N^2 - x + 1$.) In our example, we fill in the first column with three 1s and three 6s.

Now that we have filled in column 1, it's time to fill in column N with the complements of column 1. This means wherever you see a 6 in column 1, place a 1 in column 6, and vice versa. This generates the rightmost column: 1, 6, 6, 1, 6, 1.

Fill the remaining columns on the left side of the center line using the numbers already in the column, or their complements, again with the caveat that there must be equal numbers of each integer in the column. Columns on the right side of the center are filled with the complements of the left side. (You can imagine a mirror plane coming out of the paper that reflects numbers on the left with their complements on the right.)

Now let's fill square B in a manner similar to the method for the first square except that only the integers $(0, N, 2N, 3N, \ldots, (N-1)N)$ are used. (Example: For a 3×3 square, we would use the integers 0, 3, and 6.) Begin by filling each diagonal, starting from the upper left and upper right with the numbers $(0, N, 2N, 3N, \ldots, (N-1)N)$. Fill the remaining cells of the first row with integers that are already in two of them or their complements. Continue as in square A, except this time a horizontal mirror reflects all of the numbers above the square's middle with their complements below.

Finally, add corresponding numbers in A and B to get magic square C.

Similarly, the de la Hire method can be used to construct doubly even order squares. The first square A uses the numbers 1 through N arranged in a way as we just discussed. The N integers

used in the second square B are $0, N, 2N, 3N, \ldots, (N-1)N$ as discussed. Add corresponding numbers in A and B to get magic square C.

1	3	2	4
4	2	3	1
4	2	3	1
1	3	2	4

A

0	12	12	0
8	4	4	8
4	8	8	4
12	0	0	12

B

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

C

We can construct Albrecht Dürer's square from the Introduction using a similar approach.

4	3	2	1
1	2	3	4
1	2	3	4
4	3	2	1

A

12	0	0	12
4	8	8	4
8	4	4	8
0	12	12	0

B

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

C (Dürer's square)

Various authors have used variants of de la Hire's method to construct magic squares of any order. For example, the first two squares that follow (A and B) are used to generate the magic square on the right (C):

3	1	4	2	5
5	3	1	4	2
2	5	3	1	4
4	2	5	3	1
1	4	2	5	3

A

15	0	20	5	10
0	20	5	10	15
20	5	10	15	0
5	10	15	0	20
10	15	0	20	5

B

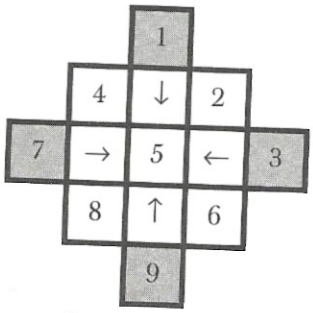
18	1	24	7	15
5	23	6	14	17
22	10	13	16	4
9	12	20	3	21
11	19	2	25	8

C

Notice that in the leftmost array A, the numbers 1 through 5 are arranged so that every number appears once and only once in every row and column and one main diagonal. The other diagonal is composed of all 3s. Notice that all the short left diagonals repeat the same number. In array B, the numbers 0, 5, 10, 15, and 20 (0, N, $2N$, $3N$, . . . , $(N-1)N$) are treated in the same way; in this case 10s are repeated along a main diagonal. Notice that all the right short diagonals repeat the same number. If these squares are superimposed and added, they produce the magic square on the right.

Stairstep Method

You may construct a magic square of odd order using the following easy recipe that involves a staircaselike assembly of cells. As an example, let's construct a 3×3 magic square. First draw a "staircase" with consecutive integers along diagonals like the following:



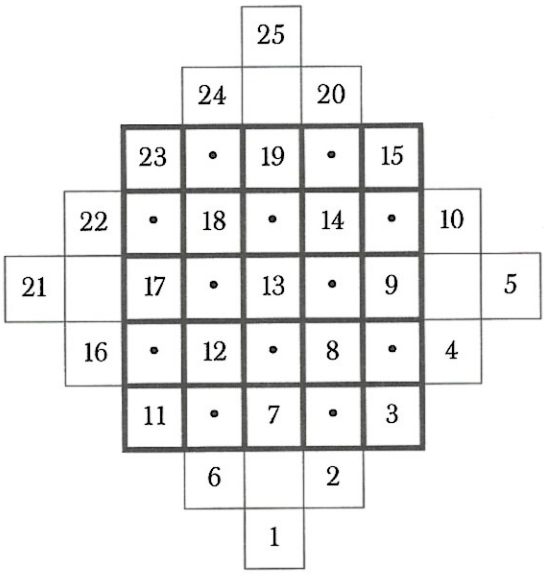
Stairstep method

The cells with 1, 3, 7, and 9 are outside the central 3×3 array. Slide the numbers in these outliers to the vacant cells furthest away along the same row or column. This produces the following magic square:

4	9	2
3	5	7
8	1	6

Third-order magic square

Here is another example of the staircase method, this time used to construct a 5×5 magic square:



Staircase before "infolding"

23	6	19	2	15
10	18	1	14	22
17	5	13	21	9
4	12	25	8	16
11	24	7	20	3

Final 5×5 magic square

This time, let's start with the 1 on the bottom cell and proceed with numbering diagonally to the right. The dots represent empty cells to be filled in a manner similar to the previous 3×3 cell example.

John Lee Fults's Triangular-Parallelogram Method

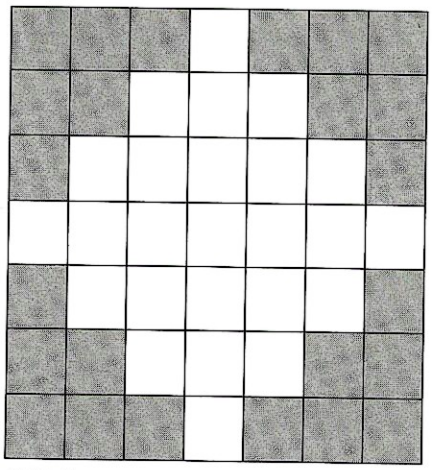
John Lee Fults, author of *Magic Squares*, invented a method that can be used to construct odd-order magic squares. Write the integers 1 through

N^2 starting at the upper left as in the following array. Next chop off four "triangular" corners of the square, which leaves an internal square that is rotated so that its corner sticks up. (In this example, the 3 sticks up.)

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Fult's method

For a 5×5 square, you chop off (L-shaped) triangles of three cells each. For a 7×7 square you chop off triangles of six cells each. In general, the corner triangles contain $(N^2 - 1)/8$ cells.



Cell-chopping in a seventh-order square

Next, you simply move each of the cut-off triangles along the main diagonal to opposites sides of the internal square. For example, in this case the upper left triangle composed of the numbers 1, 2, and 6 moves downward to the right to the position of the three dots. Notice how this creates the bottom right group (23, 6, 19, 2, 15)

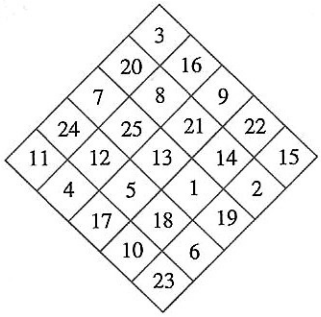


Figure 1.1
Diamond magic square.

in Figure 1.1. The triangle at the upper right composed of the numbers 4, 5, and 10 moves downward to the left. Move the other two triangles in a similar manner. When the moves are completed, you will get a diamond-shaped magic square (Fig. 1.1). Why not try this for other odd orders? If you attempt to construct a 3×3 square, the "triangles" you chop off will consist only of a single cell at each corner in the original square.

Ralph Strachey's Method

Here's a method that you can use to construct certain magic squares of singly even order.[4] As discussed, this class of squares has an order that can be divided by 2 but not divided by 4. For example, magic squares of order 6, 10, and 14 are singly even-order squares. Recall that singly even squares of order N are defined such that $N = 4x + 2$ for positive integer values of x. In the following example, $N = 6$ and $x = 1$. The Strachey method is not quite as elegant as for odd squares in the sense that the method is largely empirical. Singly even magic squares are a big challenge to construct using simple rules.

Let's describe the method generally before giving a specific example. Start by subdividing a square into four equal parts, A, B, C, and D.

A	C
D	B

Strachey's method

Eventually we will insert square arrays into *A*, *B*, *C*, and *D*; these four square arrays will be of odd order *M* each with $N^2/4$ cells. For our example case of $N = 6$, this means that each subsquare is of order $M = 3$, with nine cells.

Here's a general description. Start by using de la Loubère's method to construct a magic square with integers 1 through M^2 where $M = N/2$. Jam this magic square into *A*. Construct and jam three other magic squares into *B*, *C*, and *D*. These additional magic squares go from $M^2 + 1$ through $2M^2$, $2M^2 + 1$ through $3M^2$, and $3M^2 + 1$ through $4M^2$. You will now have a large square filled with consecutive integers 1 through N^2. If you examine this $N \times N$ square, you'll find that the sum *R* of the integers in each row is the same. The sum *C* of the integers in each column is the same, but *R* does not equal *C*. The sums D_1 and D_2 of the integers in the main diagonals are different from the column sums or row sums. By swapping a small number $(N^2/4 - N)$ of integers, you can convert the square into a magic square.

Sound complicated? Let's look at an example to clarify this mumbo-jumbo. I remind you that these singly even squares are all of order $4x + 2$. Let's choose $x = 1$, which gives us a sixth-order square. Start by dividing a square into four equal squares *A*, *B*, *C*, and *D*. Using de la Loubère's method, we use the numbers 1 through 9 to construct the upper left square in the following array:

8	1	6	26	19	24
3	5	7	21	23	25
4	9	2	22	27	20
35	28	33	17	10	15
30	32	34	12	14	16
31	36	29	13	18	11

Strachey's method

In square *B*, we use the same approach to construct a magic square using numbers 10 through 18. In square *C*, we use the same approach to construct a magic square using numbers 19 through 27. In square *D*,

we use the same approach to construct a magic square using numbers 28 through 36. Look carefully. We seem to be close to having a 6×6 magic square. Notice that at this point, all rows sum to the same number, 84. All columns sum to 111. The right main diagonal sums to 165. The left main diagonal sums to 57. We can turn this 6×6 square into a magic square by swapping just three pairs of numbers. To help you see the swaps, I've singly and doubly underlined pairs and put a pair in bold italic. Just by swapping the 8 and 35, the 4 and 31, and the 5 and 32 we get a perfect sixth-order magic square. You can make a tenth-order magic square with $(N^2/4 - N) = 15$ swaps within four 5×5 subsquares, but because the exact pattern of swapping seems to be something arbitrary, that must be memorized, I won't go into further detail here. For additional examples, see Fults.[5] Perhaps you or some other farsighted person in the future will be able to discover a method for creating this kind of square in a more systematic manner.

The Diagonal Method

The diagonal method is a quick and easy way to make doubly even magic squares. All you have to do is write down the numbers 1 through N^2 in order, starting at the upper left cell. Highlight the two main diagonals and also the main diagonals of every 4×4 nonoverlapping subsquare that composes the main square. (More precisely, the nonoverlapping subsquares are formed when the $4k \times 4k$ array is divided into k^2 4×4 subsquares.) All numbers in cells along the diagonals are swapped with skew-related cells. (Recall from the Introduction that cells placed at equal distances on opposite ends of an imaginary line through the square's center are called skew-related cells.) That's all there is to it! An example will help clarify.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Before diagonal swaps

16	2	3	13
5	11	10	8
9	7	6	2
4	14	15	1

After diagonal swaps

In this example with a 4×4 square, we first write the consecutive numbers 1 through 16. Next we examine the main diagonals and make symmetrical swaps across the center of each highlighted diagonal. This means 1 swaps with 16, 6 with 11, 13 with 4, and 10 with 7. Isn't that a great way to make a magic square? Impress your friends by using this approach on a huge square, such as the following 12×12 example.

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96
97	98	99	100	101	102	103	104	105	106	107	108
109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132
133	134	135	136	137	138	139	140	141	142	143	144

Before diagonal swaps

144	2	3	141	140	6	7	137	136	10	11	133
13	131	130	16	17	127	126	20	21	123	122	24
25	119	118	28	29	115	114	32	33	111	110	36
108	38	39	105	104	42	43	101	100	46	47	97
96	50	51	93	92	54	55	89	88	58	59	85
61	83	82	64	65	79	78	68	69	75	74	72
73	71	70	76	77	67	66	80	81	63	62	84
60	86	87	57	56	90	91	53	52	94	95	49
48	98	99	45	44	102	103	41	40	106	107	37
109	35	34	112	113	31	30	116	117	27	26	120
121	23	22	124	125	19	18	128	129	15	14	132
12	134	135	9	8	138	139	5	4	142	143	1

After diagonal swaps

This is a little bit trickier because in addition to swapping cells along the main diagonals, we also have to locate cells that are in the diagonals of all the 4×4 subsquares, one of which is shown here at upper right, and swap their values with skew-related cells. For example, 4 and 141 are swapped, as well as all other cells not already swapped in the main diagonals, such as 37 and 108, 26 and 119, and 15 and 130. When we're finished, we generate the magic square on the right with the magic constant $\mathcal{S} = N(N^2 + 1)/2 = 870$.

Dürer's Method

In the Introduction we discussed Albrecht Dürer's 4×4 magic square constructed in 1514 A.D. At that time Dürer described a general method for constructing doubly even magic squares. First, draw imaginary main diagonals through every 4×4 subsquare of the square such as in the following square on the left. (This is similar to the diagonals drawn in the previous section.) Place the number 1 in the upper left corner cell and proceed horizontally to the right with consecutive numbers, but only write the numbers down when the cells are crossed by diagonals. Continue this procedure for each row. Once you arrive at the bottom right corner cell, pretend that the number 1 falls on this square and reverse the process, proceeding horizontally to the left and filling the unfilled cells. For a fourth-order square we have the following before-and-after diagrams:

1			4
	6	7	
	10	11	
13			16

Durer's method, before

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Durer's method, after

Knight's Move Method

One elegant method[6] for constructing a variety of odd-order squares involves the knight's move in chess (which is a move two cells horizontally or vertically and then one cell at a right angle). In our application, the knight's move is constrained to two cells upward and one cell to the right. Start by placing a 1 in the center cell of the upper row. Pretend that the magic square is wrapped around a doughnut so that the top cells meet the bottom cells and right cells meet left cells, and start galloping your horse.

We'll use a fifth-order magic square for our example. The horse gallops off the square at the top and wraps around to the cell labeled 2 at the bottom. Next, the horse hops to the 3 and then 4 and 5. The next move is blocked by 1, so 6 is written below 5. The horse continues until the array is filled.

10	18	1	14	22
11	24	7	20	3
17	5	13	21	9
23	6	19	2	15
4	12	25	8	16

Knight's move method

In this square, each row, column, and main diagonal sums to 65. Also note that the sum of the two numbers in any two skew-related cells is always equal to a constant sum, 26, which is twice the value of the center square or half the magic constant. Again, as a reminder, the skew-related cells are on opposite ends of an imaginary line drawn through the center and include such pairs as (6, 20) and (23, 3). Try varying this method. For example, you should be able to produce magic squares in which the knight's move may be upward and to the left instead of to the right, or downward to the right or left.

Lozenge Method

Another method for generating odd-order magic squares was discussed by mathematician John H. Conway under the name "lozenge" method,[7] and the method appears to date back at least to the early 1900s.[8] To use this approach, create an interior diamond filled with odd numbers as in the following diagram. Start with 1 at the leftmost vertex and go diagonally upward to the right. Continue filling diagonally until the diamond is completed as shown with the bold numbers.

			2	8
18	24	5	6	12
22	3	9	15	16
1	7	13	19	25
10	11	17	23	4
14	20	21	2	8

Lozenge method

The even numbers that were missed are now added sequentially along the continuation of the diagonals, with certain wrapping rules applied when the edge of the magic square is reached. For example, consider the first diagonal 1, 3, 5. We are missing the numbers 2 and 4. To determine where these go, continue the diagonal upward and to the right. This takes us to an imaginary square above the 6. Whenever you find yourself in an imaginary square above a column, slide down. (When you are in an imaginary square to the left of a row, slide to the left.) This means the 2 slides down to the bottom row. The diagonal is continued to the right for the 4. If we continue diagonally to the right, this brings us back to our starting 1 using the wrap rule. This means we must start again with the next diagonal, which contains 7 and 9. Continue up to the right with the missing number 6. Continue off the board with an 8, which wraps to the bottom. Continue up to the right and place a 10 outside the square, which wraps to the left. Continue up and to the right. Because we have reached the starting point, we consider the next diagonal of odd numbers. The process continues until the square is filled with the missing even numbers.

The Diagonal Rule for Odd-Order Magic Cubes

Although I focus on magic squares in this chapter, I'd like to whet your appetite for constructing higher-dimensional figures such as magic cubes. A magic cube is a cube composed of N^3 consecutive numbers, where N is the order of one of the squares on the cube's faces. The numbers used in forming a magic cube are usually 1

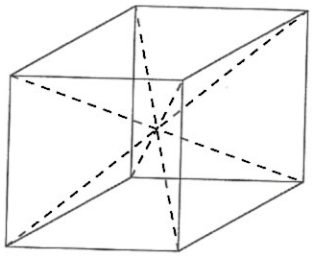


Figure 1.2
The four triagonals, or space diagonals, of a cube.

through N^3. They are arranged so that each row, each column, each main diagonal of the square cross sections, and each of the four great (space) diagonals (sometimes called "triagonals") containing N integers will add to the same sum (Fig. 1.2). (Magic cubes are thoroughly defined in chapter 2.) In 1988, John Hendricks developed a simple approach for constructing odd-order magic cubes.[9]

To create a magic cube, start by placing the number 1 at the center of the bottom layer of the cube (see Fig. 1.3). Next, begin to fill the vertical center plane as shown in Figure 1.2. You fill in the numbers on the plane just as in a magic square, proceeding diagonally downward to the right. When you run off the bottom of the plane, you reenter at the top. When you run off the right-hand side, you reenter on the left-hand side.

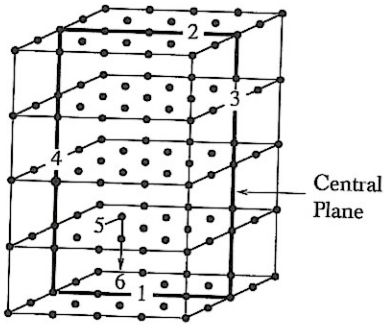


Figure 1.3
Creating a magic cube using the diagonal rule. In the cubic array, place 1 in the center of the bottom layer, then proceed diagonally down to the right along a central vertical square cross section, as in the diagonal rule for squares, to place 2, 3, 4, and 5. When you can no longer proceed, go back one layer and down one to place the number 6. This backward move is represented by the arrow from 5 to 6. Then continue diagonally and repeat the jump step as required.

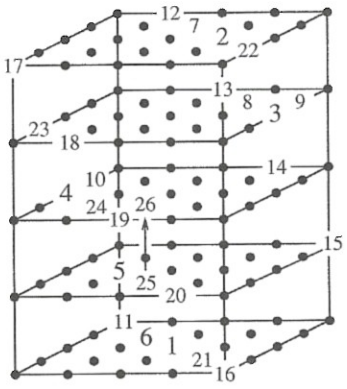


Figure 1.4

Additional steps in cube creation. You have proceeded until reaching a double impasse that occurs at 25. Move up one cell every time you have completed N^2 numbers, as indicated by the arrow. Continue moving diagonally until the cube is complete, using the "back one, down one" routine for a single impasse and the "up one" routine for a double impasse.

Each time there is a number in the way, move back one layer and down one, as shown in Figure 1.3 for the move between 5 and 6. In other words, because 1 blocks the 6, the 6 is shifted backward and down as indicated by the arrow. This motion is called a jump-step. Continue to proceed diagonally in this new plane, repeating the jump-step, one back and one down, as required. Eventually, after N planes and N^2 numbers, another impasse is reached. At this point go up one, as indicated by the arrow between 25 and 26 in Figure 1.4.

Use the "up one" procedure only if you cannot use the "back one and down one" procedure, that is, at the point where the number is divisible by N^2. Eventually, you will arrive at a fifth-order cube of magic sum 315 with the following layers:

110	12	44	71	78
98	105	7	39	66
61	93	125	2	34
29	56	88	120	22
17	49	51	83	115

Top layer of cube

79	106	13	45	72
67	99	101	8	40
35	62	94	121	3
23	30	57	89	116
111	18	50	52	84

Second layer of cube

73	80	107	14	41
36	68	100	102	9
4	31	63	95	122
117	24	26	58	90
85	112	19	46	53

Third layer of cube

42	74	76	108	15
10	37	69	96	103
123	5	32	64	91
86	118	25	27	59
54	81	113	20	47

Fourth layer of cube

11	43	75	77	109
104	6	38	70	97
92	124	1	33	65
60	87	119	21	28
48	55	82	114	16

Bottom layer of cube

Using Hendricks's approach, you can write out magic cubes of odd order as large as you want! Now that's something to impress your buddies with at the next stockholders' meeting.

Other Methods

Many other methods for constructing magic squares exist, including Arnoux's method (discovered in 1894) and Margossian's method (discovered in 1908).[10] These novel approaches have been discussed in the literature, for example in *Magic Squares* by John Lee Fults, *Magic Squares and Cubes* by W. S. Andrews, and W. W. Ball and H. S. M. Coxeter's *Mathematical Recreations and Essays.* Most writers develop their favorite methods, but the holy grail of magic squares creation would be to discover a method that would generate every possible arrangement for a square of a given size. Such a solution is probably not discoverable.

One of my favorite, little-known ways of generating one magic square from another involves matrix multiplication. In 1992, Frank E. Hruska, a Professor of Chemistry at the University of Manitoba,[11] suggested that if a third-order magic square is treated as a matrix, then cubing the matrix will produce another matrix square. The following is an example:

4	9	2
3	5	7
8	1	6

3 × 3 matrix, $S = 15$

1149	1173	1053
1029	1125	1221
1197	1077	1101

3 × 3 matrix cubed, $S = 3375$

On the left is a magic square that produces the magic square on the right when it is treated as a matrix and cubed. Recall that for matrix multiplication you don't simply cube each number in the matrix. The product C of two matrices A and B is defined by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj},$$

where j is summed over for all possible values of i and k. Here is an example to refresh yourself about matrix multiplication.

$$\begin{bmatrix} 3 & -2 & 1 \\ 2 & -1 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & 2 \\ 4 & -1 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 3(-1)+(-2)4+1(5) & 3(2)+(-2)(-1)+1(6) \\ 2(-1)+(-1)4+3(5) & 2(2)+(-1)(-1)+3(6) \end{bmatrix} = \begin{bmatrix} -6 & 14 \\ 9 & 23 \end{bmatrix}$$

The previous magic square example shows that the matrix cubed generates a magic square with magic constant 3375. If you cube the matrix again, then you arrive at a magic square with magic sum 38,443,359,375. Isn't this a great, compact way to generate magic squares with large values?

12,814,121,349	12,815,780,229	12,813,457,797
12,813,789,573	12,814,453,125	12,815,116,677
12,815,448,453	12,813,126,021	12,814,784,901

Matrix cubed again, $\mathcal{S} = 38,443,359,375$

Now, if we cube this last matrix, we get the following behemoth magic square:

Magic Square Cell Values	*Row*	*Column*
1893837622053263813460921731094 9	1	1
1893837622052825569291689052182 9	1	2
1893837622053439111128614802659 7	1	3
1893837622053351462294768266877 3	2	1
1893837622053176164627075195312 5	2	2
1893837622053000866959382123747 7	2	3
1893837622052913218125535587965 3	3	1
1893837622053526759962461338442 1	3	2
1893837622053088515793228659530 1	3	3

Matrix cubed again

The magic constant is

56,815,128,661,595,284,938,812,255,859,375

Colleague Phil Hobbs and I notice that raising any third-order magic square to any odd power seems to yield a magic square. (We used a computer to test this up to the fifteenth power.) However, simply squaring the matrix does not produce a magic square. Interestingly, all the even powers have a peculiar structure, namely, all the elements on the main diagonal are the same, and all the off-diagonal elements are the same, too. Thus, the even powers fail to be a magic square because the main diagonal sums are different from the other sums, and because the elements are not all different numbers.

It appears that you can easily demonstrate that cubing any 3×3 magic square (considered as a matrix) produces another magic square.[12] Consider that any 3×3 magic square can be represented in one of two forms. For example, one form by John Hendricks can be represented as

$y-a$	$y+a+b$	$y-b$
$y+a+b$	y	$y-a+b$
$y+b$	$y-a-b$	$y+a$

Order-3 matrix

Here the magic sum is $3y$. In the original example 3×3 square, $y = 5$, $a = 1$, and $b = 3$. The cube of this square is

$9y^3-3a^3+3ab^2$	$9y^3+3(a^2-b^2)(a+b)$	$9y^3-3a^2b+3b^3$
$9y^3+3(a^2-b^2)(a-b)$	$9y^3$	$9y^3-3(a^2-b^2)(a-b)$
$9y^3+3a^2b-3b^3$	$9y^3-3(a^2-b^2)(a+b)$	$9y^3+3a^3-3ab^2$

Matrix cubed

This result is also always a magic square with magic sum $27y^3$.

Allan Johnson, Jr., notes that you can also write 3×3 magic squares in another form:[13]

$p+s+t$	$p-2s$	$p+s-t$
$p-2t$	p	$p+2t$
$p-s+t$	$p+2s$	$p-s-t$

Order-3 matrix

When this square is treated as a matrix and cubed, we get

$9p^3+12s^2t+12st^2$	$9p^3-24s^2t$	$9p^3+12s^2t-12st^2$
$9p^3-24st^2$	$9p^3$	$9p^3+24st^2$
$9p^3-12s^2t+12st^2$	$9p^3+24s^2t$	$9p^3-12s^2t-12st^2$

Matrix cubed

This result is also always a magic square with magic sum $27p^3$.

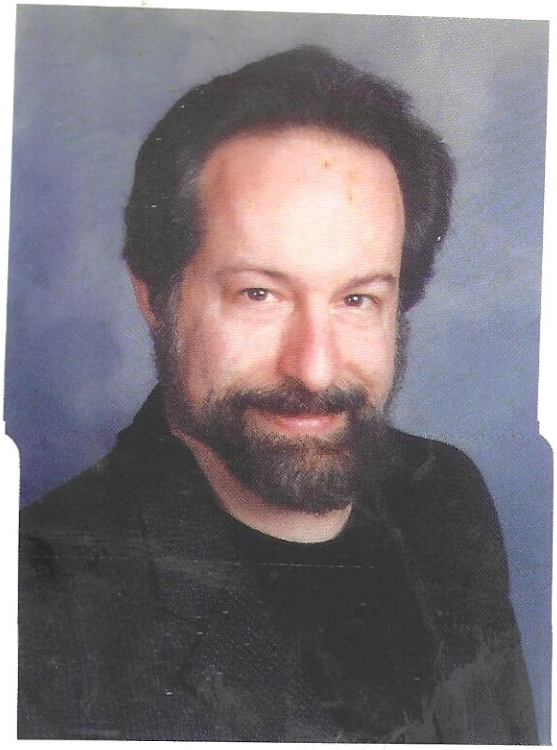
I believe the field is wide open for study with respect to cubing higher-order magic squares. Consider the following example experiment in which a 4×4 pandiagonal magic square ($\mathcal{S} = 34$) yields another magic square ($\mathcal{S} = 39{,}304$) when cubed.[14] Again, the squared matrix is not a magic square. One wonders which magic squares of arbitrary size yield magic squares when treated as a matrix and cubed. Experiments show that not all 4×4 magic squares turn into magic squares when cubed.

1	15	4	14
12	6	9	7
13	3	16	2
8	10	5	11

4×4 matrix, $\mathcal{S} = 34$

9226	10,106	10,186	9786
10,346	9626	9386	9946
9466	9866	10,426	9546
10,266	9706	9306	10,026

4×4 matrix cubed, $\mathcal{S} = 39{,}304$



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ISBN: 0-691-07041-5

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